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Inventory models for time dependent variable cost items

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INVENTORY MODELS
FOR TIME DEPENDENT
VARIABLE COST ITEMS

by

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ABSTRACT

In dealing with inventory models and systems a number of simplifying assumptions are always made. One of the usual assumptions is that the cost of the item is independent of time, or at least deterministic with respect to time.

The intent of this work is to propose a purchasing and inventory policy to deal with the situation in which the costs of the items are not deterministic and to illustrate the advantages of such a system. The basic model proposed is an extension of the conventional deterministic demand model. A computer program is presented that will calculate critical "price breaks" at each time period in the cycle. The "price breaks" are the decision variables that determine whether or not a purchase should be made in a given period. Several simulation runs are performed to illustrate the results. Investigations of the behavior of the basic model are conducted to determine the sensitivity of the model when the parameters estimated for the cost distribution are wrong.

Mathematical extensions of the model such as allowing backorders (with deterministic demand) and modifications of standard Q systems and P systems (stochastic demands) making use of price break ideas are also given. The approach used in these extensions has been to fix the item cost at its expected value and optimize relative to the standard measures. Then the cost is allowed to assume its stochastic properties and a new optimization is performed within the framework previously established.

I - INTRODUCTION

I-A Purpose and Scope

The problem considered in this thesis concerns the determination of optimal purchasing policies for an inventory system when the cost of the items in the system is variable.

In dealing with inventory models and systems a number of simplifying assumptions are always made. According to Elmaghraby [2, p. 12, 13]* the simplest inventory model - the Wilson Model - assumes the following:

(1) Continuous known rate of demand, R units/unit time.

(2) Infinite planning horizon, i.e., an infinite duration of the process.

(3) Satisfaction of all demand, i.e., no stockouts or late deliveries are permitted.

(4) Immediate delivery of replenishments, i.e., no lead time.

(5) One item in inventory, or alternatively, no interaction among the items if there are more than one.

(6) Two costs are involved in the management of stock: a) the cost of investment, assumed proportional to the average quantity in stock, and b) the cost of ordering and receiving which is assumed independent of the quantity ordered or received.

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*Numbers enclosed in brackets refer to the item number in the bibliography.

(7) Cost of the item, a constant independent of time and quantity ordered.

The work in the more sophisticated inventory models consists of relaxing one or more of the above assumptions simultaneously. Very little work, however, has been done with the time dependency in assumption seven. As mentioned before, the work presented here attempts to relax the assumption that the cost of the item is a constant independent of time and then to extend the basic model thus developed to cover additional assumptions.

The situation in the first case to be considered is the following: consider the standard deterministic inventory model (Wilson's Model), but suppose that the cost of the item fluctuates and that the probability distribution and parameters of this fluctuation have been estimated. The effects of using incorrect estimates will be studied in detail.

A possible method to attack the above problem is to use conventional decision criteria under uncertainty such as minimax, minimum regret (Savage's criteria), Bayes criteria, etc. to determine the lot size. These lot sizes will differ depending on the particular criteria used, but the total cost for the system will be approximately the same in all cases provided the differences between lot sizes are small. The reason for this is that the total cost curve when plotted against lot quantities is nearly flat around the optimum lot quantity point. For example, a 25% underestimate of the optimum lot quantity will result in less than a 4.2% increase in the total cost for the system in relation to the optimum cost and a 25% overestimate of the optimum lot quantity will cause an increase of only 2.5% in the total cost as shown by Hadley and Whitin [8, p. 36]. In addition to this, there is the risk that the cost of the item is near its maximum at the time the system is forced to buy since otherwise it would run out of stock. It is assumed in the basic model that stockouts are not allowed. Obviously, if the price variation is significant (the importance of the variance will be analyzed quantitatively in the body of the report), some method other than a deterministic model and stationary decision theory should be employed.

Two possibilities in the nature of the price variation of the inventory item stand out:

a. The price variation may follow a trend. That is, there may be random variations in any time period, but the price in a period is not independent of the price during the previous period.

b. The prices in different periods are independent of each other. This case (using various probability distributions) is the one that will be studied in detail.

At this point a hypothetical situation will best serve to illustrate the problem. Suppose that the system is one period (this period is an arbitrary time unit used as a reference, for example a day, week, etc.) away from the reorder date or reordering point and the price of the item goes down to its lower limit. Shall a purchase be made now at this minimum price or shall we wait one period and be then forced to buy at an unknown price which probably will be higher?

The answer to this question indicates one of the ways in which a solution to this problem can be obtained. Common sense tells us that if breaking points in the "price" scale and in the "time before the system is forced to reorder" scale could be calculated--i.e., buy if cost $<$ $Cost_i$ and $Time_i <$ time $< Time_{i+1}$--, it would result in a better policy than blindly buying when the system risked going out of stock. Of course, this last statement remains to be proved.

Other topics that require attention in this introduction are the following:

a. Basic material and definitions required as a prerequisite for this thesis.

b. The optimality of the models to be presented in this thesis.

c. Factors to be considered in the practical use of the models.

References that cover the background material required are the following:

--Nature of inventory systems and basic definitions - Hadley and Whitin [8, chapter 1] , Miller and Starr [13, chapter 1], Barish [21, chapter 18] , Buffa [22, chapters 15,16] .

--Problems of practical application of inventory models - Hadley and Whitin [8, chapter 9] , Miller and Starr [13, chapter 7,9].

--Probability and statistics - Hoel [23] .

An optimum purchasing policy together with an optimum inventory system is obviously desired for all the cases presented in this paper. However, the mathermatical formulation of this general problem is extremely difficult, so instead of trying to find the optimum policy, the more modest goal of trying to find the optimum policy within a subgroup of all the possible policies will be pursued. This is the approach generally taken in inventory texts. For example see Miller and Starr [13, section 35] .

The philosophy of the models presented in this thesis has been to make them as simple as possible. Generally, an accepted model for a given situation--i.e., a deterministic demand with backorders model--has been taken and extended to take advantage of the price variations. However, the basic form of the original model is preserved. This approach is shown schematically in Figure 1.

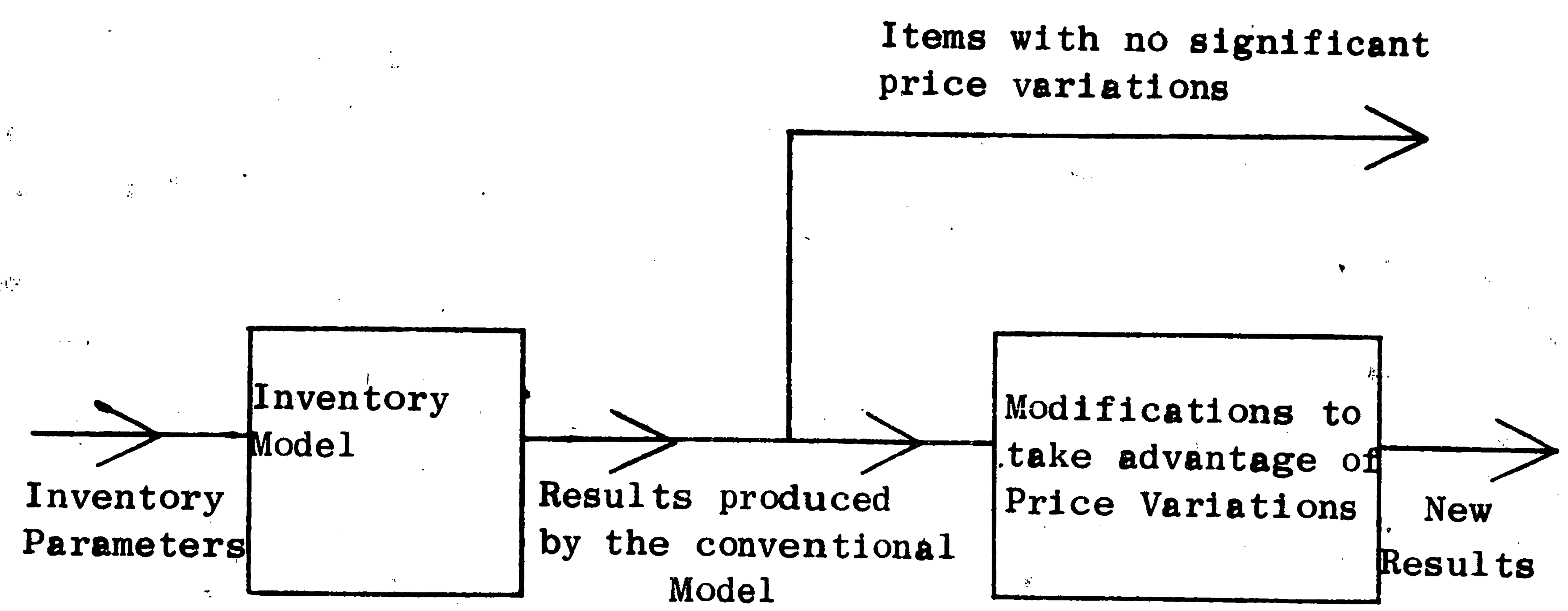


Figure 1 - Modification of Inventory Models for Variable Cost Conditions.

According to Miller & Starr [13, section 4], there are three major costs involved in the solution of inventory problems. The first major class of costs is the procurement costs. They can be of the ordering cost type or of the set-up cost type. The second class of costs is the stockage costs: the costs of carrying and of not carrying inventory. The third and last class of costs are called the systemic costs. Systemic costs can be defined best by describing their origin and nature. They are related to the amount of administrative effort required by different systems such as different amount of inspections of inventory levels, different amount of data-processing required, etc. There also may be large differences in installation and implementation costs. These costs have not been emphasized in the literature on inventory theory but they are of considerable importance in practical applications of the theory. No effort has been made in this paper to include the effects of systemic costs. The purpose here has been to introduce the models, calculate their sensitivity to mistakes in parameter estimation and to show their mathematical validity under the assumptions used. It remains for later work to test them in actual use with "real-life" data and in such a test is where the systemic costs have to be considered.

I-B Related Work

The basic idea for this paper was obtained from Miller and Starr [13, p. 142-146]. They formulate an optimal policy for selling an asset given a specific discrete probability distribution for the price variation and without considering any holding costs.

The problem in Miller and Starr was based on a paper by Morris [14] . In this paper Morris considers an optimum single procurement strategy, but he doesn't include the ordering and holding costs. Morris's results are obtained by using expected values and the results given here for the basic model, except for the holding cost which he didn't consider, agree with his. Doing further research in this area, it was found that the part of Morris's paper of interest here was based on previous work done by Simon [17 , chapter 14]. Simon presents an optimum strategy to sell a house. The three papers mentioned above have two items in common that set them apart from the basic model in this thesis. One of them is that holding costs are not explicitly included and the other is that the problem is not considered in the context of inventory theory.

A recent paper by Veinott [18] on the status of mathematical inventory theory states that the only works in this area are the paper by Morris already mentioned and three others, given here as references [3], [10], [11] which will be discussed below.

Fabian, Fisher, Sasieni and Yardeni [3] consider the problem of purchasing raw material in a fluctuating market. They employ strictly dynamic programming for the solution of the theoretical problem and assume that the following data are available for decision making: a) the existing inventory, b) the current market price, c) the cost of holding inventory and the cost of shortage, and d) the probability-density functions for future price and demand of the raw material. The model is essentially of the "P type" [13, section 38] since purchasing decisions are made at fixed intervals. The results of the theoretical model are complicated and several assumptions are made to arrive at a "practical model." This "practical model" makes use of price forecasting. Although the assumptions in this article are not the same as the ones proposed here, it is a very interesting paper since it reports on the only known attempt to use an inventory model with price fluctuations in a "real life" situation.

Karlin [10] approaches the problem from the point of view of selling an asset. Part of his mathematical analysis gives results similar to the ones reached in the basic model in this paper (by a different method), but Karlin does not propose any inventory model.

Karlin and Carr [11] consider the problem of prices and optimal inventory theory but the assumptions are completely different from the ones made in this paper and the models are developed within the framework of price theory. Their work does not apply to the situation considered here.

In a recent paper, Kaminsky [9] develops a constant order quantity model for a single commodity inventory system in a market subject to stochastic price variation. He assumes that two purchase prices are available to the stock controller. A regular purchase price is assumed to be in effect at any point in time and an opportunity purchase price is assumed to arise at random points in time. In the development of the model unit demands and opportunity prices are assumed to arise according to independent Poisson processes.

Finally there is a very interesting paper written by Sakaguchi [15] that Veinott [18] does not mention, probably because the title of Sakaguchi's paper does not bear any relationship to the problem of buying or selling a commodity or to inventory theory. Sakaguchi attacks the problem within the framework of sequential sampling design but he develops "optimum stopping rules" that correspond closely to the rules obtained for the basic model presented in this thesis. He also considers the case in which the random variables may have partially unknown distributions; for example, the normal distribution with unit variance and an unknown mean. It is assumed that an a priori probability distribution for the value of the mean is available. The method presented extracts and accumulates information about the unknown true parameter of the population distribution from successive observations.

II - BASIC MODEL: DETERMINISTIC DEMAND AND NO BACKORDERS

II-A Description of the Problem

The problem consists of studying the inventory policies of an item whose total yearly cost is given by

$$Y = NC + AX + \frac{NC}{2X} I + \frac{N}{2X} T \tag{1}$$

where

Y = Yearly cost

N = Demand/year

C = Price of the item

A = Ordering cost per lot

X = Number of lots per year

I = Annual interest in percent/100

T = Inventory holding cost (other than interest) per unit per year

If this were a completely deterministic model, it would only be necessary to minimize the yearly cost with respect to the number of lots purchased per year. The optimal solution would be

$$X = \sqrt{\frac{NCI + NT}{2A}} \tag{2}$$

The ordering period, the order quantity, and the total cost per year can be easily obtained after having the optimum number of lots to be purchased per year.

Suppose, however, that the price of the item varies and that the limits of this variation are known (or that they can be estimated). Let the lower price limit be C_L and the upper price limit be C_U. Furthermore, as discussed earlier, assume that the price fluctuations from day to day are independent of each other.

A few words must also be said with respect to the time lag between order and delivery. Since all the quantities in this initial model--except the price of the item--are considered to be deterministic, the time lag or procurement lead time does not present any problems. It is merely required that an order be placed a fixed number of periods before the time it would have been placed if the procurement lead time had been zero.

The optimum policy for deterministic conditions mentioned before does not consider the NC term in the cost equation (the total cost of buying the items). This is the term, however, that accounts for most of the variation in total cost when purchases are made at different unit costs.

A policy that takes into account the NC term will be considered here. This policy assumes a fixed order quantity and also that the system is forced to buy before the stock gets down to a certain level. Suppose that the expected price for the item can be obtained, or at least estimated and call it $\overline{C}$. Then the deterministic policy can be represented graphically as shown in Figure 2 where Q is the order quantity and P is the ordering period determined directly from (2)* when $\overline{C}$ is used as the unit cost.

*Numbers in parenthesis refer to the equation number in the text.

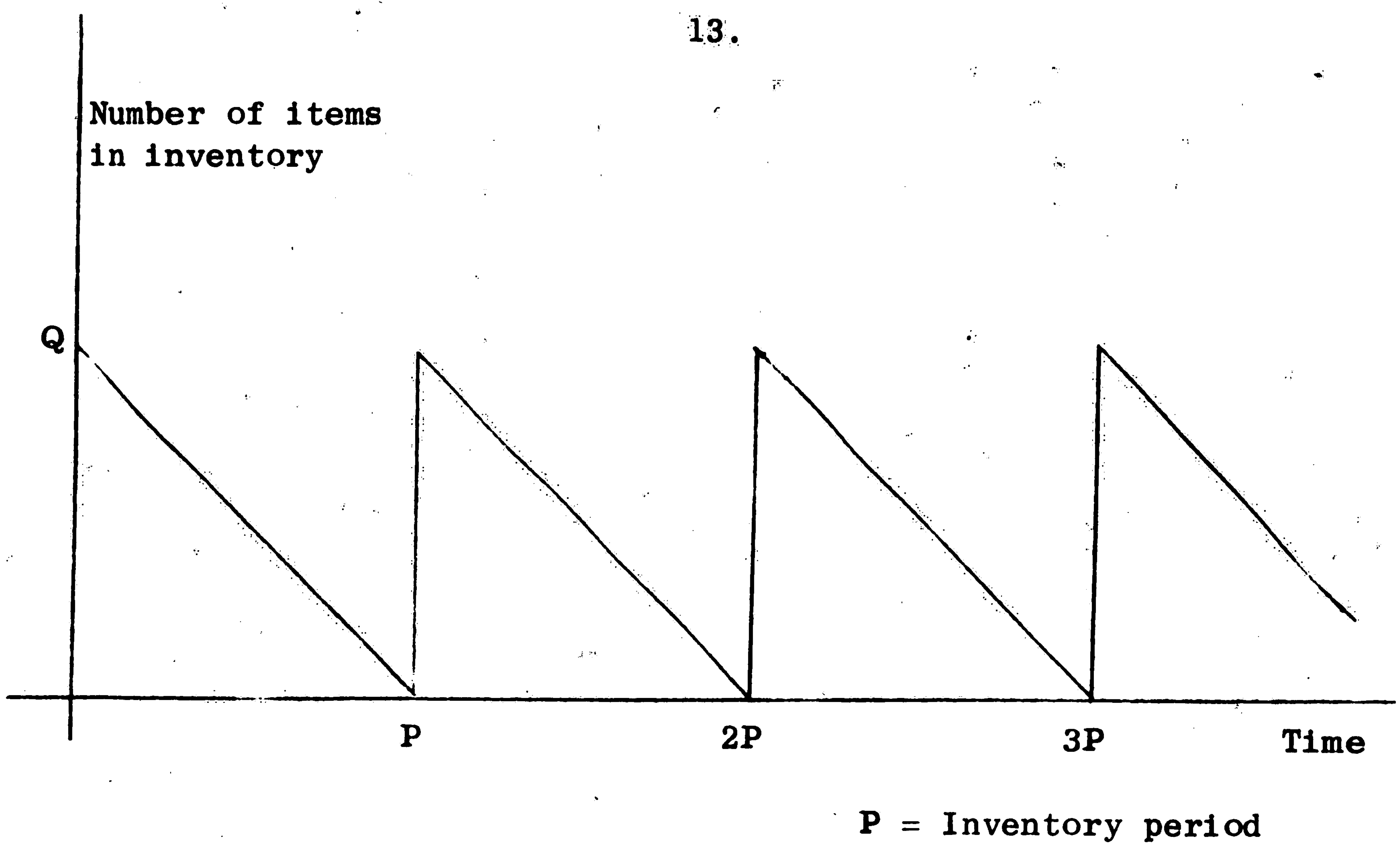


Figure 2 - Graphical Representation of a Deterministic Inventory System.

The policy proposed uses this same ordering quantity Q. There must be one and only one order placed in each inventory period. The difference is that now the inventory controller does not wait until he is forced to buy at whatever price is effective on the day the system runs out of stock (or the day a purchase is required considering the lead time involved). This policy orders at any point during the cycle depending on the price quoted for a particular time period (hour, day, week, etc.) and how many periods there are left before a purchase has to be made.

The model then will be guided by the following rules: buy quantity Q if the price during this time period is $C < C_i$ and there are "i" additional time periods before a "forced" purchase has to be made. This policy is illustrated in Figure 3. Under the assumptions given, the maximum amount of stock on hand is 2Q (but this will not occur in every cycle) instead of Q.

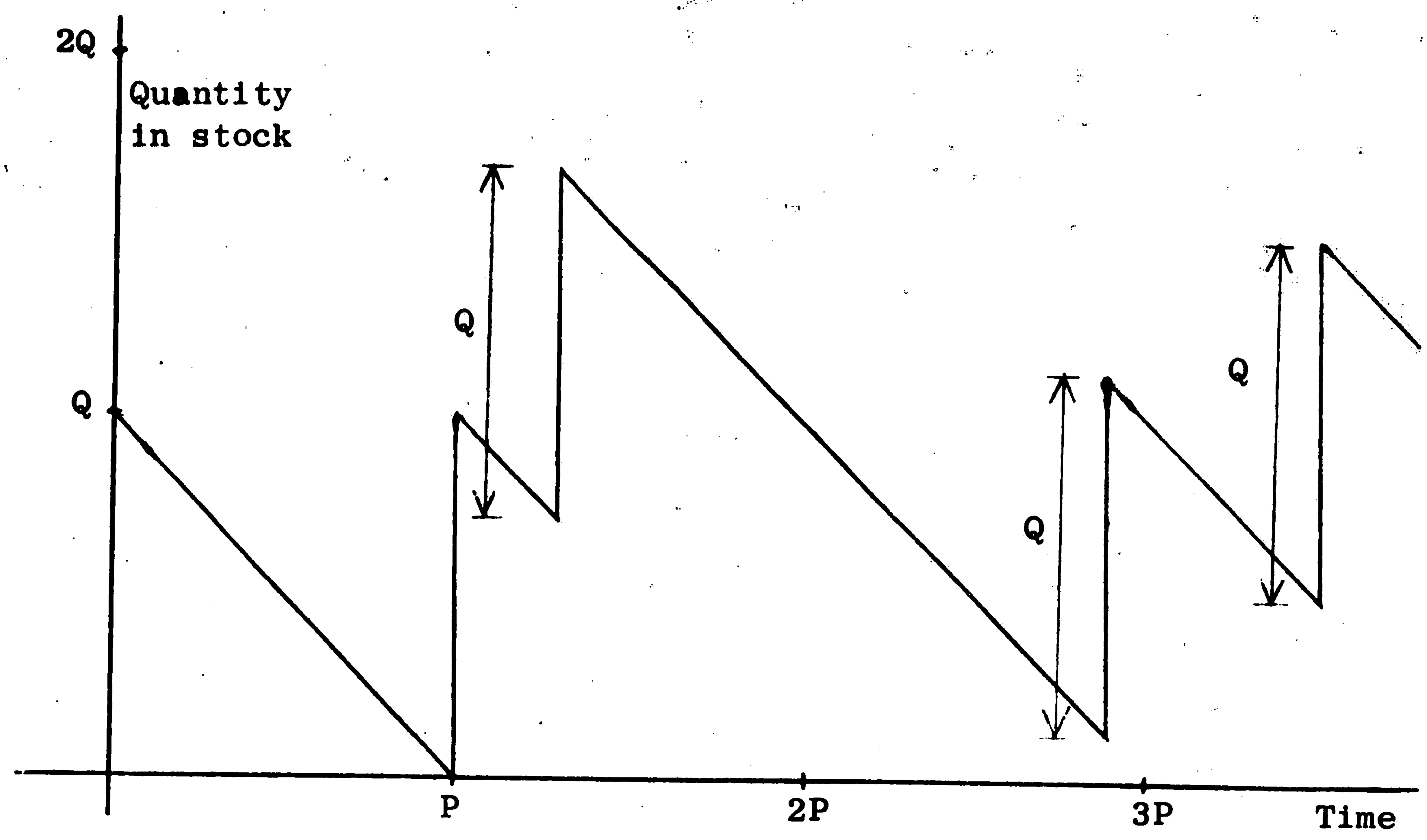


Figure 3 - Graphical Representation of an Inventory System with Stochastic Prices.

II-B Mathematical Model

The four terms in the total cost equation (1) will be considered first. One of them, NC, is determined by the price paid for the items. The second term depends on the number of lots bought per year and the last two terms depend on the average inventory on hand. It will be proved later (p. 22) that even if the ordering cost is neglected, the optimum policy under the assumptions set forth in the preceding section will not consist of buying partial quantities several times during an inventory cycle. This is based on minimizing the expected cost. Due to this reason, the ordering cost term will not be considered in the discussion concerning the determination of optimum price breaks. The ordering cost will simply be added once per cycle to the total cost equation.

The total cost equation for the "price break" model is

$$Y + NC + AX + \frac{NC}{2X} I + \frac{N}{2X} T$$

where the meaning of the different symbols has been previously explained. Using the expected price $\overline{C}$, the optimal number of lots per year can be found to be

$$X = \sqrt{\frac{N\overline{C}I + NT}{2A}} .$$

From this, it is easy to find the order quantity since $Q = N/X$ and also the average ordering period in days from $P = 365/X$. Of course, the average ordering period can be found in any other time basis as desired. It will be considered that a new price is quoted every time period and that the price in any given time period is independent of the prices quoted in previous time periods.

The probability distribution of the price variation is needed for the mathematical analysis of the problem. If the uniform distribution is assumed and the price of the item turns out to have a different distribution (but the mean is correct), the expected total cost resulting from the use of the "price break" system with the wrong distribution will be less than the one resulting from any of the decision criteria under uncertainty. This will be illustrated in part D of this section.

The decision criteria will be based on expected values (this is also done by Miller and Starr [13, section 41] and by Morris [14]). That is, the cost of buying during any time period (including the extra inventory holding cost for buying before necessary) will be compared with the cost if no purchase is made during that time period and an optimum policy is followed thereafter. The price breaks that minimize the expected cost when looking at the system from the "total horizon" point of view, that is, considering one complete inventory cycle will be determined. The following additional symbols will be required:

q = price quotation for any time period

q_k = critical price for time period k

k = number of complete periods since the beginning of the cycle

$f(q)$ = density function of the price during any one period

L = total number of periods available to make a decision

= also the inventory cycle calculated using average price

P_k = probability that the price in period k will be below q_k

i = inventory holding cost in dollars per unit in stock per time period (calculated using T, I and the average cost)

n = number of time periods left before a purchase has to be made ($n = L - k$)

The probability that the price in period k will be below q_k is

$$P_k = \int_0^{q_k} f(q)\, dq \qquad (3)$$

The average price paid for the item if it is bought in period k plus carrying charges until the last period in the cycle times the probability that a purchase is in fact made in period k is given by

$$e_k = \int_0^{q_k} (q + ni)\, f(q)\, dq \qquad (4)$$

This expression deserves an additional explanation.

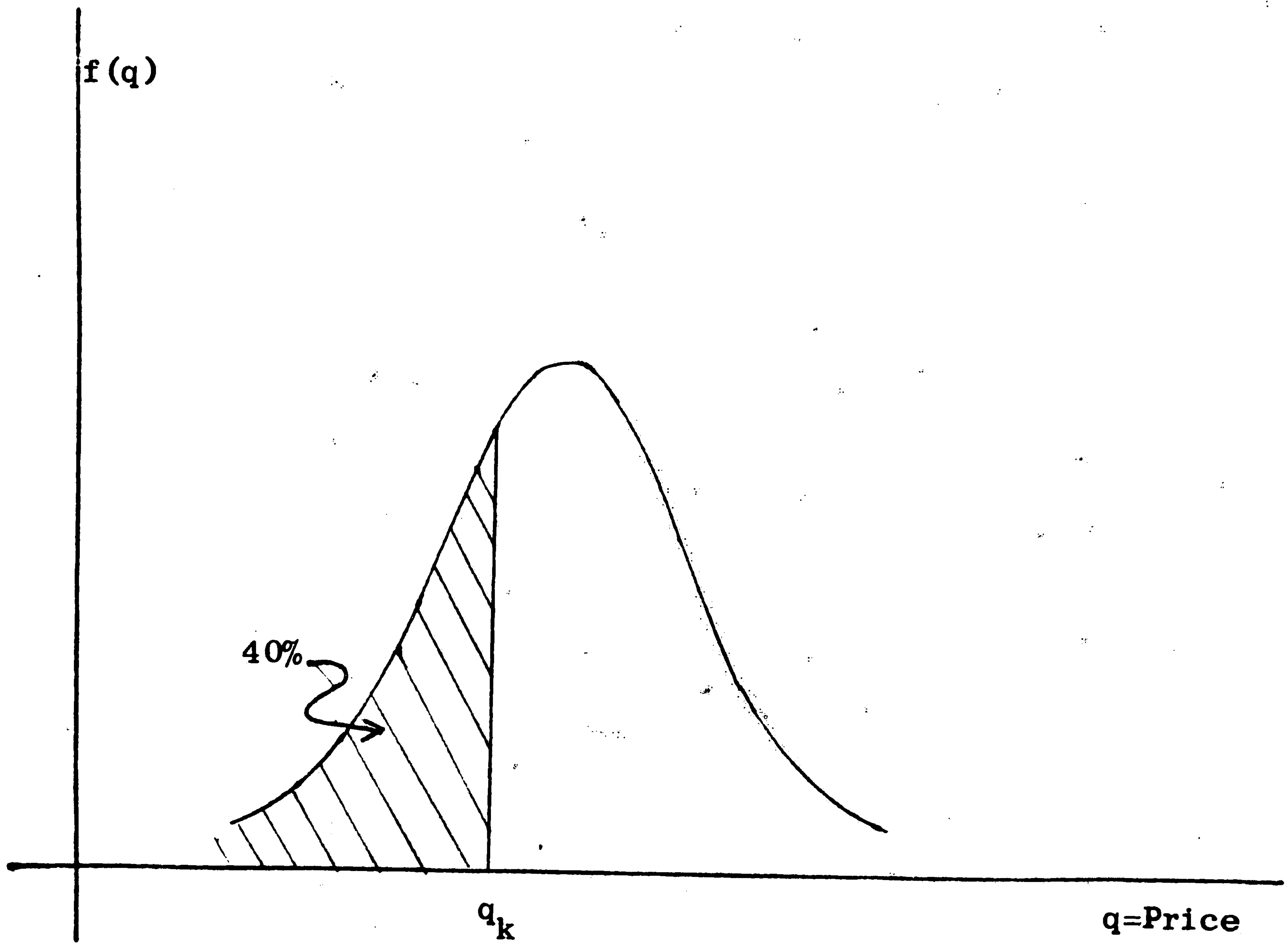


Figure 4 - Probability Distribution of the Price.

Suppose, as shown in Figure 4, that the probability that the price in a period is below q_k is 40%, that is $P(q < q_k) = .4$. The probability distribution $f(q)$ has to be divided over .4 in order to obtain a new probability distribution with upper limit q_k. The average for this new distribution with the price projected to the end of the cycle is

$$\int_o^{q_k} (q + ni) \frac{f(q)}{.4} dq \qquad (5)$$

Since e_k has been defined as the product of this average times the probability that a purchase is made in this period--which is .4--the following result is obtained

$$e_k = .4 \int_o^{q_k} (q + ni) \frac{f(q)}{.4} dq = \int_o^{q_k} (q + ni) f(q)dq \qquad (6)$$

The expected total expenditure (with holding costs added as required) if the decision of not buying during this time period when there are n periods left before the end of the inventory cycle is

$$EH_{(L-n)} = e_{L-n+1} + \sum_{v=L-n+1}^{L-1} e_{v+1} \left\{ \prod_{j=L-n+1}^{v} (1-P_j) \right\} \qquad (7)$$

This is the expected cost if the system follows the optimum path during the last (n - 1) time periods in the inventory cycle. The above expression can also be given in terms of k, the days elapsed since the beginning of the inventory cycle

$$EH_k = e_{k+1} + \sum_{v=k+1}^{L-1} e_{v+1} \left\{ \prod_{j=k+1}^{v} (1-P_j) \right\} \qquad (8)$$

This equation can also be expressed in a different form using the principle of optimality from dynamic programming [20]. This principle states that an optimal policy has the property that, whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with respect to the state resulting from the first decision. The following recurrent relation is equivalent to equation (8)

$$EH_k = e_{k+1} \ (1-P_{k+1})EH_{k+1}$$

$$k = 0,1,2,\ldots,L-1 \tag{9}$$

It will be instructive to check equation (8) when there are 3 periods left and a decision is made to wait, that is, to follow the optimum policy when there are two periods left and thereafter. Under this condition n is 3 and $k = L-3$. It will be assumed in this example that the time periods are days. The total expected cost should be the average price in day L-2 (projected as usual to the end of the cycle) times the probability that a purchase is made on that day or e_{L-2}, plus a similar quantity for day L-1 times the probability that a purchase was not made on day L-2 (otherwise a purchase couldn't be made on day L-1), plus a similar quantity for day L times the probability that a purchase was not made on day L-2 or day L-1. The above explanation can be expressed mathematically as follows:

$$EH_{L-3} = e_{L-2} + e_{L-1} \ (1-P_{L-2}) + e_L \ (1-P_{L-2})(1-P_{L-1}) \tag{10}$$

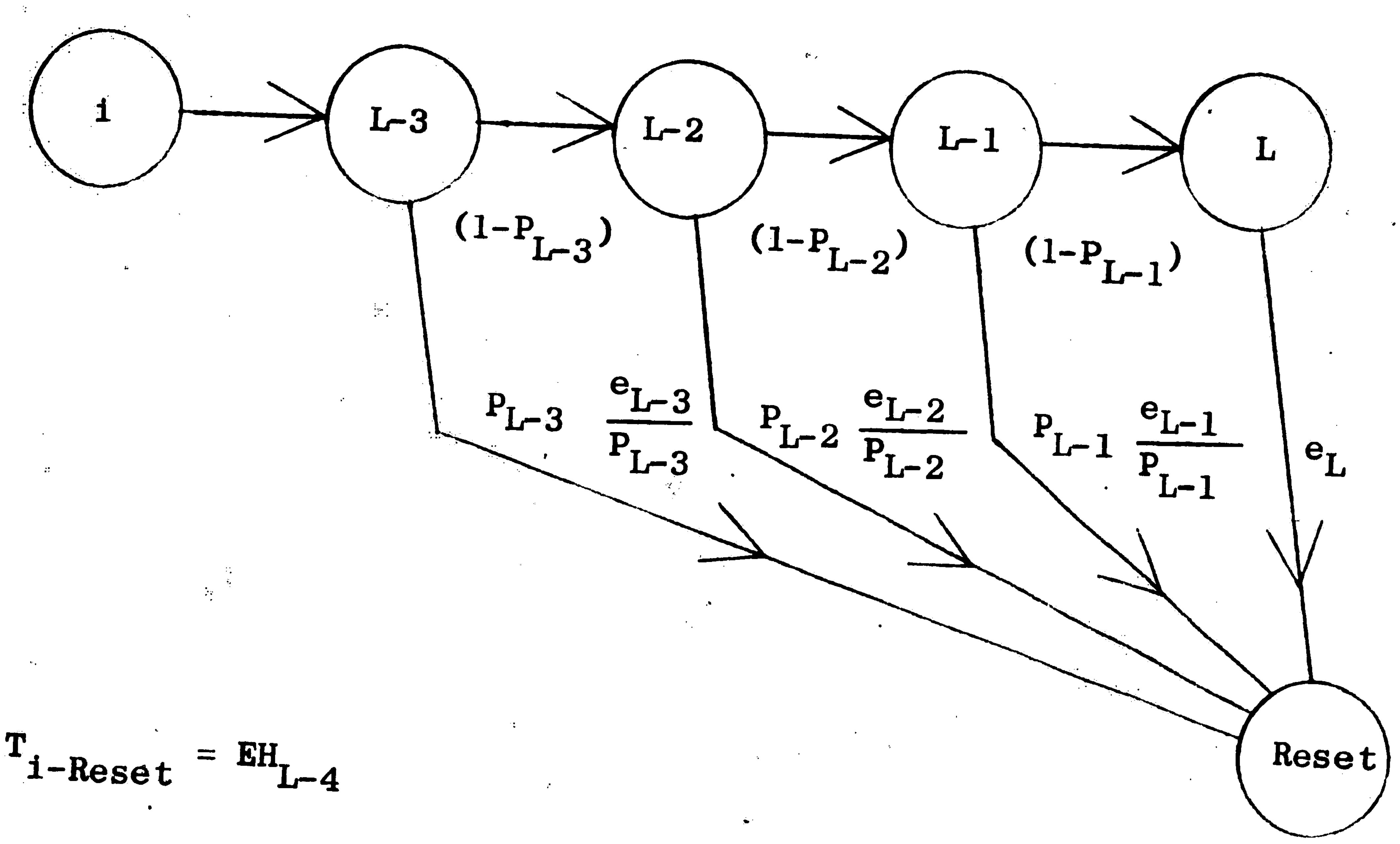
This result is the same as the one obtained by letting n = 3 in the general formula for $EH_{(L-n)}$.

The expected cost of the item including holding costs can be represented graphically if each decision period is considered as a state of the system. Then each state is represented by a node in the network. For example, suppose it is desired to represent EH_{L-4}. According to equation (7)

$$EH_{L-4} = e_{L-3} + \sum_{i=L-3}^{L-1} e_{i+1} \left\{ \prod_{j=L-3}^{i} (1-P_j) \right\} EH_{L-4} = e_{L-3} + \qquad (11)$$

$$e_{L-2}\,(1-P_{L-3}) + e_{L-1}\,(1-P_{L-3})(1-P_{L-2}) + e_L\,(1-P_{L-3})(1-P_{L-2})(1-P_{L-1})$$

This value of EH_{L-4} is given by the transmittance from node "i" to node "reset" of the graph in Figure 5.



$T_{i\text{-Reset}} = EH_{L-4}$

Figure 5 - Graphical Representation of EH_{L-n}.

The optimum policy (using expected values as the decision criteria) in period k is to buy if it can't be expected to do better by waiting, that is

buy if $q \leq q_k$ (12)

where $q_k = EH_k - (L-k)i$ (13)

or expressed in terms of the days left

$$q_{L-n} = EH_{L-n} - ni \tag{14}$$

Notice that the price quotation in any period has to be projected to the last period in the inventory cycle by adding the holding costs for the remaining periods since this is the way in which the expected cost EH_{L-n} has been calculated.

This last result will now be obtained formally by minimizing the total expected cost over the complete horizon, that is, over the complete inventory cycle. In this case $k = 0$, $m = L$ and the expected cost is

$$EH_o' = e_1 + \sum_{v=1}^{L-1} e_{v+1} \left\{ \prod_{j=1}^{v} (1-P_j) \right\} \tag{15}$$

$$\frac{\partial EH_o}{\partial q_k} = (q_k + ni)\, f(q) \left\{ \prod_{j=1}^{k-1} (1-P_j) \right\} + \left[\sum_{v=k}^{L-1} e_{v+1} \left\{ \prod_{j=1}^{v} (1-P_j) \right\} (-1)\, f(q) \right] \Big/ (1-P_k) \tag{16}$$

The first term is obtained from the derivative of e_k when $v + 1 = k$ and the second term takes care of all cases when $v \geq k$ (obtained from the derivative of the $(1-P_k)$ term). Letting the last expression equal to 0 the following result is obtained

$$q_k + ni = \sum_{v=k}^{L-1} e_{v+1} \left\{ \prod_{j=k+1}^{v} (1-P_j) \right\}$$

$$q_k + ni = e_{k+1} + \sum_{v=k+1}^{L-1} e_{v+1} \left\{ \prod_{j=k+1}^{v} (1-P_j) \right\}$$

$$q_k + ni = EH_k$$

$$q_k = EH_k - ni$$

This confirms the previous result.

The total cost per year can now be calculated in the usual way, except that EH_o is used in place of $\overline{C}$.

$$Y = N(EH_o) + AX + \frac{N(EH_o)I}{2X} + \frac{N}{2X}T \tag{17}$$

It was mentioned at the beginning of this section that the optimal policy will not consist of buying partial quantities several times during an inventory cycle even if the ordering costs are neglected. The proof of this statement and the notation required follows:

$EH_{L-(n+1)}$ - Expected cost per unit if optimum decisions are made over n periods. Note that this <u>is</u> consistent with the previous definition of EH_{L-n}.

$Q(q,n)$ - Amount to be purchased in any one period as a function of the price quoted during that period and the number of periods left to make a decision.

Note that $0 \leq Q\ (q,n) \leq Q_o$

Q_o - Total amount to be purchased during the inventory cycle.

$q, L, n, f(q), i$ - The meaning of these terms is the same as before.

Then

$$EH_{L-(n+1)} = \text{Expected Value of} \left[\frac{[Q(q,n)](q+ni) + [Q_o - Q(q,n)]EH_{L-n}}{Q_o} \right] \tag{18}$$

The expected value of a function according to Hoel [23, page 134] is defined as follows

$$E\left[h(x)\right] = \int_{-\infty}^{\infty} h(x)\ f\ (x)\ dx \tag{19}$$

$$EH_{L-(n+1)} = \int_{o}^{\infty} \left\{ \frac{[Q(q,n)](q+ni) + [Q_o - Q(q,n)]EH_{L-n}}{Q_o} \right\} f(q)dq \tag{20}$$

$$EH_{L-(n+1)} = \frac{1}{Q_o} \int_{o}^{\infty} \left[(q+ni) - EH_{L-n} \right] Q(q,n)\ f\ (q)dq + EH_{L-n} \tag{21}$$

The objective is to minimize equation (21) and due to restrictions in the values that q and $Q(q,n)$ can assume ($q > 0$, $0 \leq Q(q,n) \leq Q_o$), this is equivalent to minimizing the following integrand

$$\left[(q+ni) - EH_{L-n} \right] Q(q,n) \tag{22}$$

The solution then is given by

$$Q(q,n) = 0 \text{ if}(q + ni) > EH_{L-n}$$

$$Q(q,n) = Q_o \text{ if}(q + ni) < EH_{L-n}$$

It has been proved therefore that the optimal policy consists of buying in any one period all the requirements for the inventory cycle or not buying at all.

The computation of the price breaks for an item making use of equations (3), (4), (7) or (8) or (9), (12) and (13) is a very tedious operation. For this reason a computer program in Fortran IV (IBM 1130 mode) that calculates the expected costs, price breaks, and the expected total cost assuming a uniform price distribution is given in Appendix I. This program gives punched cards output in addition to printed output so that they can be used as input to the simulation programs described later.

II-C Numerical Example

The following example is presented to illustrate the method explained in the preceding section and the need for a computer program in any practical application.

Suppose it is desired to find the optimum inventory and purchasing policy (within the limits of the subgroup of possibilities considered by the model) for the following item:

T	= Misc. inventory holding cost (other than interest) per unit per year	\$145.00
N	= Number of units required per year	700
I	= Interest	20%
A	= Ordering cost per lot	\$100.00
f(q)	= Uniform distribution with limits as follows	$\frac{1}{200}$
C_L	= Lower price limit	\$1000.00

C_U = Upper price limit $1200.00

With this information, it is possible to find the optimum number of lots per year given by X

$$X = \left[\frac{N(I)(C_L + C_U)/2 + N(T)}{2A}\right]^{\frac{1}{2}}$$

$$X = \left[\frac{700(.2)(1000 + 1200)/2 + 700(145)}{2(100)}\right]^{\frac{1}{2}}$$

X = 35.74 lots/year

$$\text{AIP} = \text{Average Inventory Cycle} = \frac{365}{35.74} = 10.21 \text{ days}$$

It will be considered in this example that prices are quoted daily and no special allowance will be made for holidays. The time periods will be days in this case. Obviously, it is very easy to take care of any modification in the two assumptions above.

The planning horizon is 10 days. The inventory charge on a daily basis is approximately

$$i = \frac{I(C_L + C_U)/2 + T}{365} = \frac{.2(1000 + 1200)/2 + 145}{365} \quad (23)$$

i = $1.0000/day

The expected price during the last day is merely the average price:

day 10 $e_{10} = \dfrac{1000 + 1200}{2} = \$1100.00$

day 9 $EH_{(10-1)} = EH_9 = 1100.00$

$q_9 = EH_9 - 1\ (i) = 1100.00-1.00 = \$1099.00$

on day 9, that is, one day before the deadline buy if

$q \leq 1099.00$

$$P_9 = \int_{1000.00}^{1099.00} \frac{1}{200}\,dq = \frac{99}{200} = 0.4950$$

$$e_9 = \int_{1000.00}^{1099.00} \frac{(q + 1.00)}{200}dq = \frac{1}{200}\left(\frac{q^2}{2} + 1.00q\right)\Bigg|_{1000.00}^{1099.00}$$

$$e_9 \approx 519.99$$

day 8

$$EH_8 = 519.99 + (1. - .4950)1100$$
$$= 519.99 + 555.50 = \$1075.49$$

$$q_8 = EH_8 - 2(i) = 1075.49\text{-}2(1.00) = \$1073.49$$

on day 8, that is, two days before the deadline buy if

$q \leq 1073.49$

$$P_8 = \int_{1000.00}^{1073.49} \frac{1}{200}\,dq = \frac{73.49}{200} = .3674$$

$$e_8 = \int_{1000.00}^{1073.49} \frac{q + 2(1.00)}{200}\,dq = \frac{1}{200}\left[\frac{q^2}{2} + 2(1.00)q\right]\Bigg|_{1000.00}^{1073.49}$$

$$e_8 = 381.72$$

day 7

$$EH_7 = 381.72 + (1. - .3674)519.99 + (1. - .3674)(1 - .4950)1100$$
$$= 1061.99$$

$q_7 = EH_7 - 3(i) = 1061.99-3(1.00) = 1058.99$

on day 7, buy if $q \leq \$1058.99$

It is necessary to proceed until the values for day 1 and also EH_0 which represents the expected cost of the item (including the extra inventory holding cost) are found.

The sample output in Appendix I-C gives the complete results for the example above.

II-D Results Obtained using Conventional Decision Criteria

A possible alternative is the use of inventory models based on several decision criteria under uncertainty instead of the model based on the formulas given in section II-B.

Although the use of the following criteria, as explained in the Introduction, is not recommended in this case, they will be described briefly below.

Minimax Criteria According to this criterion each of the available strategies should be considered and the worst possible outcome for every one of them determined. Since the inventory problem deals with costs, the worst that could happen for each strategy is the largest cost which might be expected if that strategy were selected. The strategy which offers the minimum of such maximums should be selected according to the minimax criterion. Essentially, the policy consists of assuming that the worst possible conditions will occur and then to select a policy that will make this worst condition as good as possible (for additional details see Miller and Starr [13, section 19]). From a total cost point of view, the worst condition will occur if the cost of the item is a maximum, that is, if it is C_U. The optimum policy is then given by

$$X \text{ minimax} = \sqrt{\frac{NC_U I + NT}{2A}} \tag{24}$$

Minimum Regret Criteria The criterion of minimum regret was first suggested by Savage. Regret is defined as the difference in the results obtained by following the selected policy and the results that could have been obtained if the policy best suited to the events that actually occurred had been used. Savage suggested that this regret should be minimized. The policy under this criterion can be obtained as follows: let TC_{SL} be the total cost with the policy selected and assuming the effective price is C_L; TC_L be the total cost if the optimum policy assuming that the price of the item was going to be C_L (minimum price) had been followed; TC_{SU} be the total cost with the policy selected and assuming the effective price is C_U; TC_U be the total cost if the optimum policy had been followed assuming that the price was going to be C_U (max. price).

The minimum regret strategy will occur at a point equidistant from the two extremes, that is, when

$$TC_{SL} - TC_L = TC_{SU} - TC_U \tag{25}$$

or

$$NC_L + AX_R + \frac{NC_L}{2X_R} I + \frac{N}{2X_R} T - NC_L - AX_L - \frac{NC_L}{2X_L} I - \frac{N}{2X_L} T =$$

$$NC_U + AX_R + \frac{NC_U}{2X_R} I + \frac{N}{2X_R} T - NC_U - AX_U - \frac{NC_U}{2X_U} I - \frac{N}{2X_U} T$$

$$X_R = \frac{NI(C_U - C_L)}{2AX_U - 2AX_L + (NC_UI)/X_U + (NT)/X_U - (NC_LI)/X_L - (NT)/X_L}$$

$$X_R = \frac{NI(C_U - C_L)}{2A\sqrt{(NC_UI + NT)/2A} - 2A\sqrt{(NC_LI + NT)/2A} + \frac{NC_UI + NT}{\sqrt{\frac{NC_UI + NT}{2A}}} - \frac{NC_LI + NT}{\sqrt{\frac{NC_LI + NT}{2A}}}}$$

Use was made above of the fact that

$$X_L = \sqrt{(NC_LI + NT)/2A} \qquad X_U = \sqrt{(NC_UI + NT)/2A}$$

Simplifying the last expression for X_R, the following result is obtained

$$X_{Regret} = \frac{NI(C_U - C_L)}{2\sqrt{2A(NC_UI + NT)} - 2\sqrt{2A(NC_LI + NT)}} \tag{26}$$

Bayes Criteria Bayes stated that if one is really ignorant of the probability of occurrence of the various conditions, then it should be assumed that all values within the allowable range are equally likely to occur. Actually this assumption converts a problem under uncertainty into a problem under risk. Using this criteria, the optimum number of lots per year is

$$X_{Bayes} = \sqrt{\frac{NI(C_L + C_U)/2 + NT}{2A}} \tag{27}$$

Three very similar computer programs were used to simulate the inventory system under the assumption that the probability distribution for the cost of the item is uniform, normal, and triangular. In all three cases, it was considered that the actual cost distribution was unknown at the time the price breaks were calculated and the same theoretical price breaks, assuming a uniform distribution, were used.

The purpose here is to show that even if a uniform distribution for the cost is assumed and it turns out to be a different one, a lower yearly cost would be obtained than if one of the decision criteria under uncertainty had been used.

The simulation programs and a flow chart for one of the programs are given in Appendix II.

The programs provide as an output the day in the cycle in which the purchase was made, the actual price paid, and the total inventory cost for the cycle in question. This information is provided for each cycle in the simulation run. In addition a yearly cost is given for each year being simulated and an average cost per year is calculated based on the total number of years considered. The time period used in the programs is days, but this can be easily modified as required. The average yearly costs given by each of the programs for a set of numbers selected at random is given in Figures 6 - 8.

Bayes Criteria	$811,587.85
Expected Cost Using Price Breaks System and Assuming Uniform Distribution of Prices	$737,037.89
Minimum Regret Criteria	$811,587.85
Minimax Criteria	$811,594.87
Average Cost Obtained by Simulation Using Price Breaks System	$733,717.65

Figure 6 - Results of Simulation. Uniform Distribution of Prices.

Bayes Criteria	$811,587.85
Minimum Regret Criteria	$811,587.85
Minimax Criteria	$811,594.87
Average Cost Obtained by Simulation Using Price Breaks System	$743,821.05

Figure 7 - Results of Simulation. Normal Distribution of Prices.

Bayes Criteria	$855,234.76
Minimum Regret Criteria	$855,234.91
Minimax Criteria	$855,235.82
Average Cost Obtained by Simulation Using Price Breaks System	$800,713.85

Figure 8 - Results of Simulation. Increasing Triangular Distribution of Prices.

By changing only four statements in the program for the simulation of an increasing triangular distribution, it can be used to simulate a decreasing triangular distribution. The new results are shown in Figure 9.

Bayes Criteria	$767,940.95
Minimum Regret Criteria	$767,940.79
Minimax Criteria	$767,953.93
Average Cost Obtained by Simulation Using Price Breaks System	$727,383.85

Figure 9 - Results of Simulation. Decreasing Triangular Distribution of Price.

It is easy to see that the price break system has an advantage in all cases. Generalizations based in these results and others will be made later in this paper.

III SENSITIVITY ANALYSIS AND INVESTIGATION OF THE BEHAVIOR OF THE BASIC MODEL

Several investigations of the basic model are conducted in this section. These are not the only things that could be investigated, but the methods used are representative. Also, this type of investigation could be made about any of the more complicated models presented in section IV and V.

Various computer programs are presented in the appendices. All these programs use one basic program (given in appendix I) and its adaptation to specific uses was rather simple.

III-A Determination of Confidence Limits for the Use of the Model

An attempt is made in this section to answer the following question: How much can the actual mean cost shift (as compared to the one used in the calculation of price breaks) before the policy ceases to be an improvement over simply buying when it is needed?

Let the estimated mean of the price distribution be $\bar{C}$. Let the average expected cost using the price breaks and assuming the estimated mean is correct be $EH_o(\bar{C}, \bar{C})$. The first parameter inside the parenthesis indicates the quantity used to calculate the price breaks while the second parameter indicates the actual mean of the price distribution. Then obviously $EH_o(\bar{C}, \bar{C}) < \bar{C}$.

Suppose that the actual mean is not $\bar{C}$, but rather $\bar{C}_1$. Furthermore suppose that $\bar{C}_1 > \bar{C}$. Then it will always be true that $EH_o(\bar{C}, \bar{C}_1) \leq \bar{C}_1$.

On the other hand suppose that the actual mean is $\bar{C}_2$ and that $\bar{C}_2 < \bar{C}$. The price breaks will now be met more often than before and $EH_o(\bar{C}, \bar{C}_2)$ will be smaller than $EH_o(\bar{C}, \bar{C})$. For this reason if $EH_o(\bar{C}, \bar{C}) \leq \bar{C}_2$, that is, if the actual mean is equal to or larger than the expected cost using the price breaks and the wrong mean estimate, the use of the price break model produces better results than the use of no model at all.

The condition that remains to be investigated occurs when $\bar{C}_2 < EH_o(\bar{C}, \bar{C})$. It is known that for $\bar{C}_2 < \bar{C}$ then $EH_o(\bar{C}, \bar{C}_2) < EH_o(\bar{C}, \bar{C})$, but this difference has to be quantified.

Let the price breaks obtained from the distribution with mean $\bar{C}$ be $q_1, q_2, q_3, \ldots, q_{L-1}$. Then

$$EH_o(\bar{C}, \bar{C}_2) = e_1 + \sum_{v=1}^{L-1} e_{v+1} \left\{ \prod_{j=1}^{v} (1-P_j) \right\} \qquad (28)$$

where e_i and P_i are obtained by using the price distribution with mean $\bar{C}_2$ and the price breaks were calculated from the distribution with mean $\bar{C}$. Equation (28) can be expanded to the following form

$$EH_o(\bar{C}, \bar{C}_2) = \int_0^{q_1} \left[q + (L-1)i \right] f_{\bar{C}_2}(q)dq +$$

$$(1-P_{1C_2}) \int_0^{q_2} \left[q + (L-2)i \right] f_{C_2}(q)dq + \ldots +$$

$$(1-P_{1C_2})(1-P_{2C_2}) \ldots (1-P_{(L-1)C_2}) \int_0^{\infty} q \, f_{C_2}(q)dq \qquad (29)$$

where $P_{1C_2} = \int_0^{q_1} f_{C_2}(q)dq$

$\vdots$

$P_{kC_2} = \int_0^{q_k} f_{C_2}(q)dq$

If the price distribution is uniform, then equation (29) can be reduced to the following

$$EH_o(\bar{C},\bar{C}_2) = \frac{1}{f_{C_2}(q)}\left[\int_{q_h}^{q_1} [q + (L-1)i]\, dq + \right.$$

$$\left.\sum_{v=1}^{L-1}\left\{\left[\int_{q_h}^{q_{v+1}} [q + (L-v-1)i]\, dq\right]\left[\prod_{j=1}^{v} (1-P_{jC_2})\right]\right\}\right] \tag{30}$$

In this expression q_h and q_L are the low and high limits respectively of the uniform distribution $f_{C_2}(q)$ while all the other q_k's were calculated from the distribution with $\bar{C}$ as the mean.

The critical point being sought will occur when a distribution $f_{C_2}(q)$ is found such that $\bar{C}_2 = EH_o(\bar{C}, \bar{C}_2)$. If it is assumed that the distribution with mean $\bar{C}_2$ has the same range as the one with mean $\bar{C}$, then equation (30) can be solved uniquely.

Let q_h be the unknown, then $q_L = q_h + R$ and $\bar{C}_2 = q_h + \frac{R}{2}$.

Now the value of q_h that will make $\bar{C}_2 = EH_o(\bar{C}, \bar{C}_2)$ can be found. From equation (30)

$$q_h + \frac{R}{2} = \frac{1}{R}\left\{(q_1^2 - q_h^2)/2 + (L-1)i(q_1-q_h) + (1-P_{1C_2})\left[(q_2^2 - q_h^2)/2 + \right.\right.$$
$$\left.(L-2)i(q_2 - q_h)\right] +$$

$$\ldots + (1-P_{1C_2})(1-P_{2C_2})\ldots(1-P_{(L-1)C_2}) \left[\frac{(q_h + R)^2 - q_h^2}{2}\right]\Bigg\} \qquad (31)$$

The only unknown in equation (31) is q_h, but a difficulty arises since

$$P_{kC_2} = \int_o^{q_k} f_{C_2}(q)dq = \int_{q_h}^{q_k} \frac{1}{R}\, dq = \frac{q_k - q_h}{R} \qquad (32)$$

Then equation (31) is an L^{th} order equation in q_h and can only be evaluated by iterative methods.

The use of Newton's Method was considered. It was not found practical, however, due to difficulties in the evaluation of the derivative of equation (31) with respect to q_h. The following procedure was used instead:

1. Start the iterations with $\bar{C}_2 = EH_o(\bar{C}, \bar{C})$, that is, $q_h = EH_o(\bar{C}, \bar{C}) - \frac{R}{2}$.

2. A new expected cost is found using the above value for q_h. Let this expected cost be EH_{o1}. The results will be related in the following way

$$EH_{o1} < \bar{C}_{21} = EH_o(\bar{C}, \bar{C})$$

   where $\bar{C}_{21}$ means the mean of the price distribution used in the first iteration.

3. Now let $\bar{C}_{22} = EH_{o1}$ and find the new expected cost which will be called EH_{o2}. ($EH_{o2} < \bar{C}_{22} = EH_{o1} < EH_o(\bar{C}, \bar{C}) = \bar{C}_{21}$).

4. Proceed in the same form until $EH_{ok} \approx \bar{C}_{2k}$. At this point the desired value of $\bar{C}_2$ has been found.

A computer program that performs the above process for the case of a uniform distribution of price is given in appendix III. The flow chart for the program and the results of one problem are also given.

III-B Deviation from Optimality - Error in the Estimated Mean Cost per Item

The objective in this section is to consider the effects of using the wrong mean in the calculation of the price breaks.

The method followed is to change the mean of the price distribution by a percentage of the standard deviation, but maintaining the same variance. The procedure is as follows:

1. First the price breaks and expected cost are calculated for a given mean. These will be called the initial price breaks and initial expected cost.
2. The mean is increased or decreased by a percentage of the variance and the new expected item cost is calculated using the price breaks model with the new distribution and the initial price breaks. The expected item cost is also calculated using price breaks determined from the new distribution.
3. Step 2 is repeated for as many points as desired.

Figure 10 shows three curves giving the variation in expected item cost for changes in the estimated mean from [initial mean - .50 (std dev.)] to [initial mean + .50 (std dev.)]. The input parameters were those used in appendix I. Curve "a" gives the mean cost without using the price break model, curve "b" gives the expected

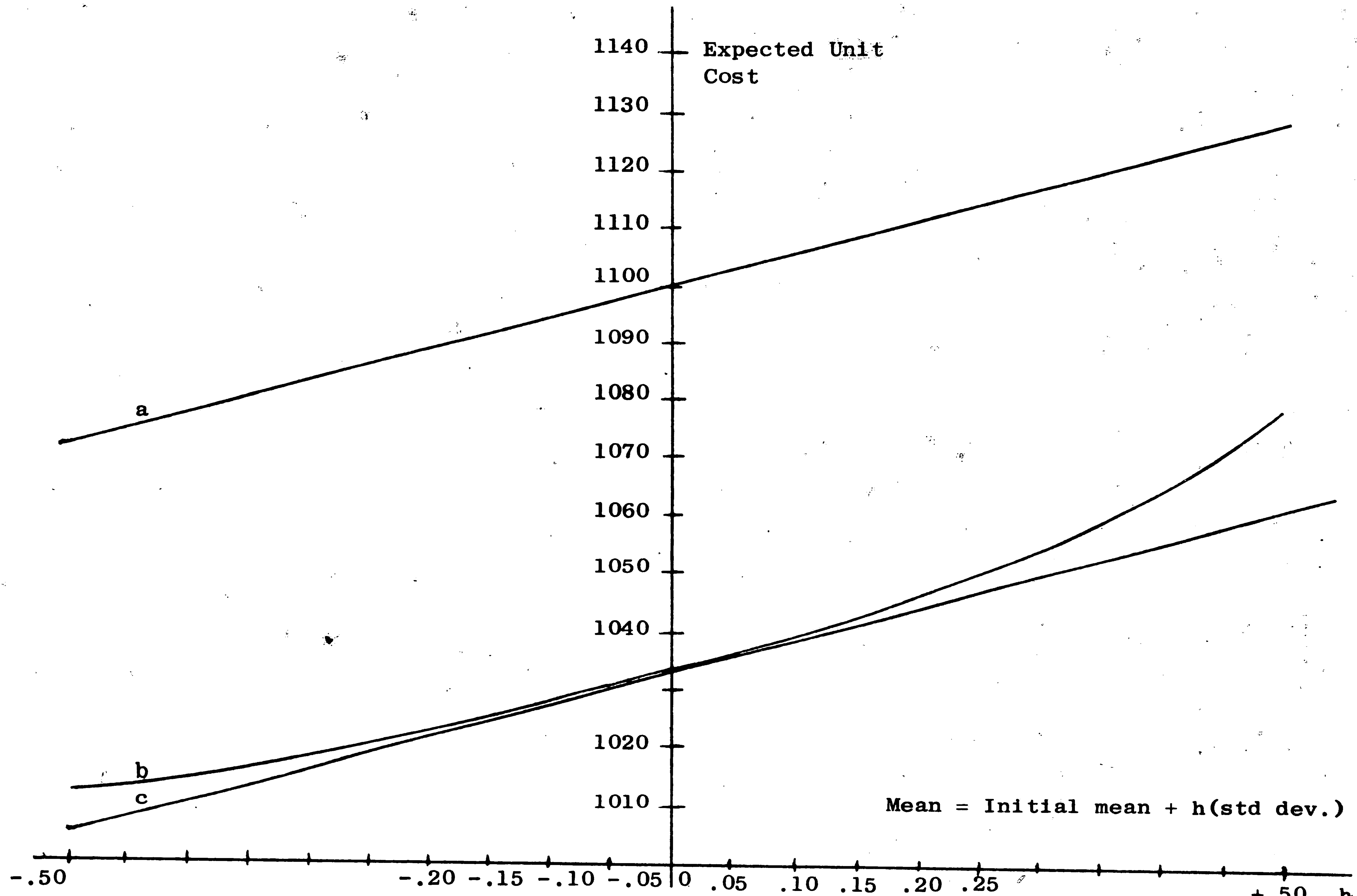


Figure 10. Sensitivity Analysis - Error in the Estimated Mean Price

item cost using the initial price breaks, and curve "c" shows the expected item cost using the optimum price breaks.

Six other set of curves were plotted for different values of the variance of the price distribution, the holding cost per decision period and the length of the decision cycle. The shape of the curves was in all cases similar to the curves in Figure 10.

It has been assumed above that the decision period or horizon has not changed. This implies that the order period and consequently the order quantity has not changed following the changes in the mean of the price distribution. This assumption is justified at least during one or two cycles following a sudden change in the mean price. A computer program that calculates the points required for curves b,c in Figure 10 is given in appendix 4-A. This program assumes that the price distribution is uniform.

III-C <u>Deviations from Optimality</u> - <u>Error in the Price Breaks Used</u>

This section is very similar to section III-B. In section III-B the effects of using the wrong price mean which in turn caused the use of wrong price breaks were studied. The objective here is to consider the direct effects of using wrong price breaks.

A sensitivity curve giving the expected item cost as a function of the deviation of the price breaks from optimality is shown in Figure 11. The deviation of the price breaks from optimality is given as a percentage of the standard deviation of the price distribution. The problem used here is again the problem given in appendix I. The computer program for this calculation is given in appendix 4-B.

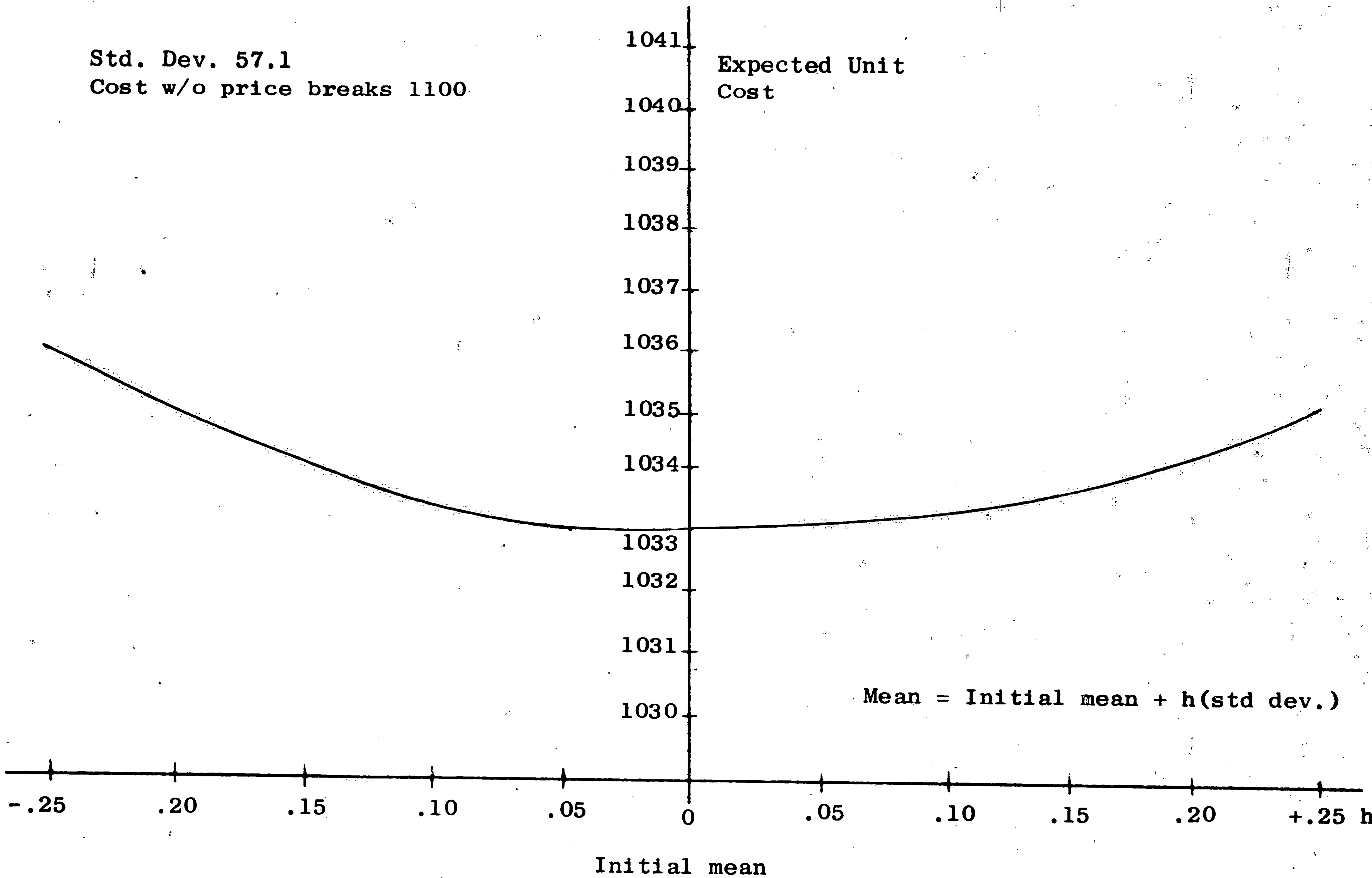


Figure 11. Sensitivity Analysis - Error in the Price Breaks

III-D Deviation from Optimality - Error in the Cost Probability Function Assumed

In this section the effect of using the wrong probability distribution to describe the cost of the item is studied. It is assumed that the mean and variance used are correct.

Initially, suppose that the price breaks were calculated assuming that the distribution was uniform and that the distribution turned out to be normal, however. The following two expressions are the only ones that have to be calculated in a form different from the one used in the computer program in Appendix I:

$$e_k = \int_{-\infty}^{q_k} (q + ni)f(q)dq$$

$$P_k = \int_{o}^{q_k} f(q)dq$$

To be able to calculate these quantities when f(q) is a normal distribution, two sets of values have to be read into the computer.

One of the sets gives the probabilities, in terms of standard deviations, of having a value less than or equal to a certain quantity. For example

$$\text{Prob} = \int_{z}^{\infty} \phi(z)dz = \int_{-\infty}^{z} \phi(z)dz \tag{33}$$

This set is made of 300 numbers giving the probabilities for $z = 0.01$ to $z = 3.00$. The second set is also made of 300 numbers and gives the ordinate of the probability curve at different values of z.

Let $\phi(z)$ be the ordinate at point z and $\Phi(z)$ be the complementary cumulative distribution. Then

$$e_k = \int_{-\infty}^{q_k} (q + ni)f(q)dq = \int_{-\infty}^{q_k} qf(q)dq + ni \int_{-\infty}^{q_k} f(q)dq$$

$$e_k = \bar{q} - \left[\sigma \phi \left(\frac{q_k - \bar{q}}{\sigma}\right) + \bar{q}\, \Phi \left(\frac{q_k - \bar{q}}{\sigma}\right) \right] + ni\, P_k$$

$$e_k = \bar{q} - \left[\sigma \phi \left(\frac{q_k - \bar{q}}{\sigma}\right) + \bar{q} \quad (1-P_k) \right] + ni\, P_k$$

$$e_k = \bar{q} \quad P_k - \sigma \phi \left(\frac{q_k - \bar{q}}{\sigma}\right) + ni\, P_k \tag{34}$$

This is the expression used in the program in Appendix VI to evaluate e_k.

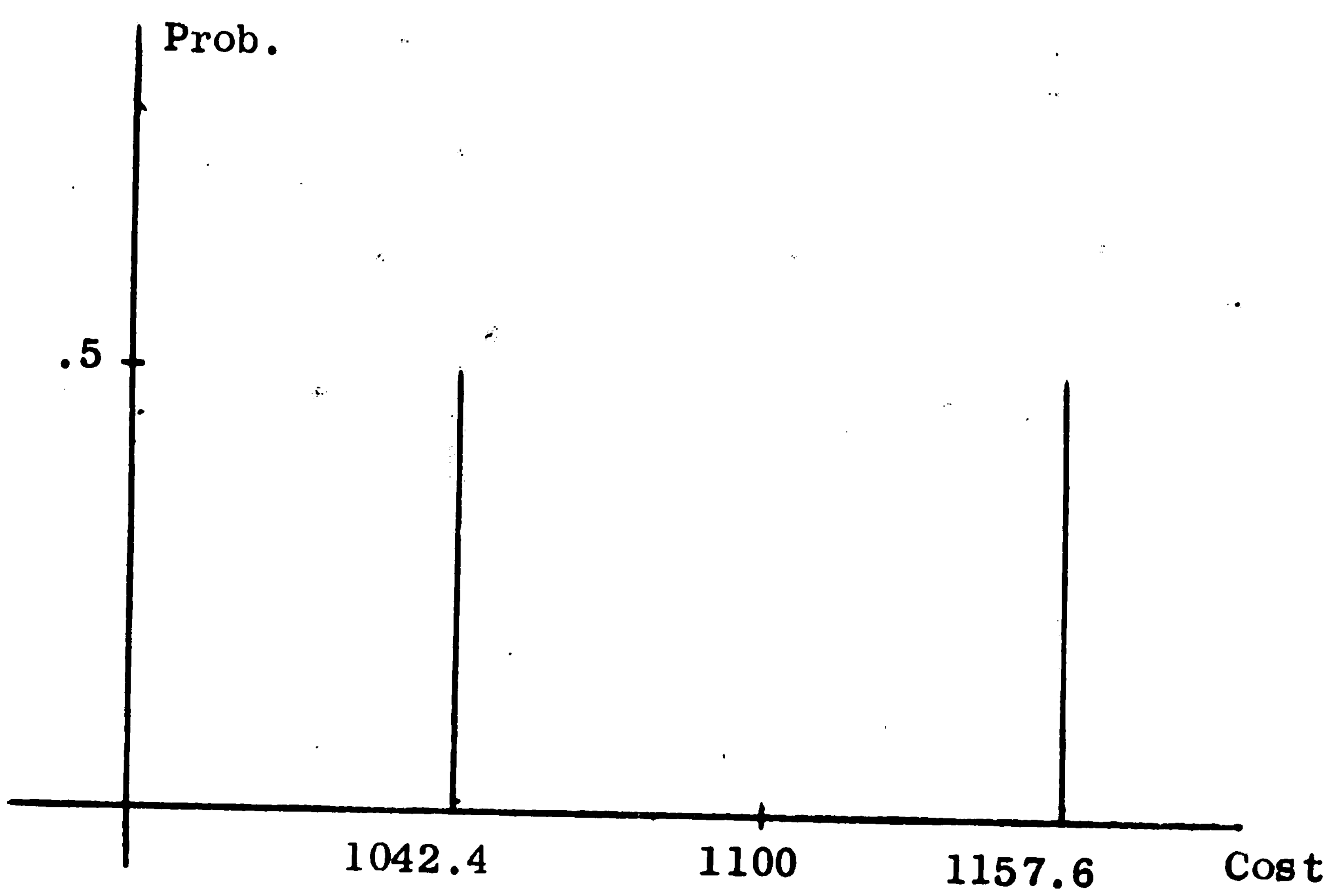
A program using the normal distribution to begin with, was also prepared. The purpose was to be able to determine the deviation from optimality caused by assuming the wrong distribution. Both programs, and flow charts for each, are given in Appendix 5. The results obtained using as input the problem described in Appendix I are given in Figure 12.

The same method used here can be followed for other distributions. As an illustration, the extreme case of a discrete function with only two possible values and the same mean and variance as the distribution of Appendix I was also considered. This distribution is shown in Figure 13. The results obtained with this distribution are shown in Figure 14.

The results in this section reinforce the results obtained with the simulations in Chapter II. It is clear that even if the distribution of costs is not known, it is advantageous to assume a reasonable one and proceed with the use of the price break model.

Item cost for uniform distribution of costs	$1,033.04
Item cost for normal distribution of costs	1,031.08
Item cost for normal distribution of costs using price breaks obtained assuming uniform distribution	1,046.00
Item costs if no price breaks had been used	1,100.00

Figure 12 - Error in the Cost Probability Function Assumed (Uniform, Normal).



Mean 1100., Standard Deviation 57.6

Figure 13 - Dichotomous Distribution

Item cost for uniform distribution of costs	$1,033.04
Item cost for dichotomous distribution of costs	1,048.02
Item cost for dichotomous distribution of costs using price breaks obtained assuming uniform distribution	1,049.00
Item cost if no price breaks had been used	1,100.00

Figure 14 - Error in the Cost Probability Function Assumed (Uniform, Dichotomous)

III-E Importance of the Variance of the Cost Distribution

The objective in this section is to show that the expected item cost when the price break model is used is very sensitive to the variance of the cost distribution.

The result mentioned above is not surprising. Obviously, if the variance of the cost distribution is zero, the cost or price of the item is constant and no savings can be obtained using the model. On the other hand if the cost has a distribution with a large variance, the use of the model eliminates the possibility of buying at the high prices during most periods (all except the last one) in the ordering cycle.

The variation of the expected unit cost for the problem in Appendix I when the standard deviation changes from 0 to 25% of the mean is shown in Figure 15. The savings as a percent of the mean cost is also given. The points in Figure 15 can be obtained by running the program in Appendix 1 several times, or by adding a loop to it and obtaining from an additional data card the values of the standard deviation at which an expected unit cost is desired.

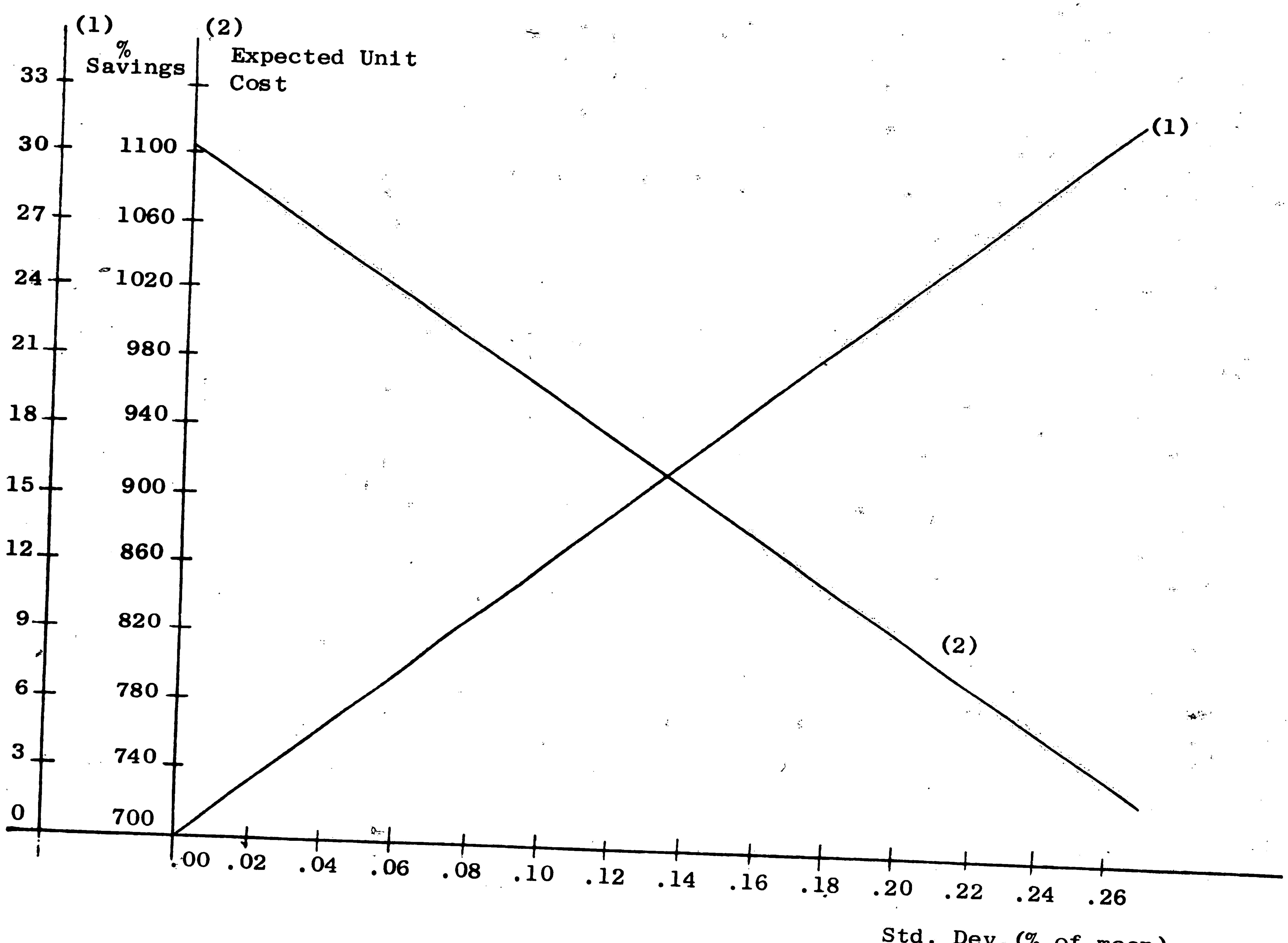


Figure 15. Variation in Expected Unit Cost as a Function of the Variance

IV. EXTENSIONS OF THE BASIC MODEL: DETERMINISTIC DEMAND

Some of the restrictions imposed on the model of Chapter II are eliminated in this chapter. The condition of deterministic demand, however, is maintained here. Two models allowing stochastic demands will be studied in Chapter V.

IV-A Different Price Distributions in Various Time Periods

It was assumed in the basic model that the price distribution of the item remained constant for all periods. If this distribution changes from period to period, and this change is known in advance, there is not any conceptual problem. The only thing that is required is that the appropriate distribution $f(q)$ be used in each period in formulas (3) and (4). That is, $f(q)$ is now a function of time also. Of course, the above reasoning assumes that the optimum order quantity and ordering cycle have been previously determined.

IV-B Conditions Required for Buying More than Once per Inventory Cycle

It was proved in Chapter II that it is never optimum to make several purchases in one inventory cycle adding up to Q_o, the optimum lot quantity. This result was arrived at using the decision criteria of maximum expectation.

There is one possibility, however, that has not been considered. Suppose that a purchase for an amount Q_o has already been made in one inventory cycle, for instance cycle "m". According to the model in Chapter II there is then enough stock on hand to last until the end of cycle m + 1. The question now is the following: Are there any conditions that would make it economical to buy again another quantity Q_o during cycle m? The answer to the above question is yes. This is simply another instance in which a cost trade-off can be used to advantage.

If a second purchase is not made during cycle m, then a purchase would have to be made in cycle m + 1. The expected item cost for buying during cycle m + 1 converted to the end of cycle m + 1 according to formula (15) is

$$EH_o = e_1 + \sum_{v=1}^{L-1} e_{v+1} \left\{ \prod_{j=1}^{v} (1-P_j) \right\}$$

The condition that must be considered now is a price quotation during cycle m (after having bought once already in cycle m) which after being projected to the end of cycle m + 1 is still less than EH_o. The procedure to obtain the price breaks <u>for the second purchase in a cycle</u> is given in the next paragraph.

Consider the last day or day L in cycle m. Buy again if $q < q_L^*$ where q_L^* indicates the price break on day L required for a second purchase in the cycle and is given by

$$q_L^* = EH_o - (L + 0)\, i$$

The other formulas for day L are

$$P_L^* = \int_o^{q_L^*} f(q)dq$$

$$e_L^* = \int_0^{q_L^*} [q + (L+0)i]f(q)dq$$

The symbols have the same meaning as in Chapter II except for the asterisk which indicates the requirements for a second purchase in the same cycle. The general formulas for this situation are then the following

$$P_k^* = \int_0^{q_k^*} f(q)dq \tag{35}$$

$$e_k^* = \int_0^{q_k^*} \left[q + (n+L)i\right] f(q)dq \tag{36}$$

$$EH_{(L-n)}^* = e_{(L-n+1)}^* + \sum_{v=L-n+1}^{L-1} e_{v+1}^* \left\{ \prod_{j=L-n+1}^{v} (1-P_j^*) \right\}$$

$$+ EH_o \left\{ \prod_{j=L-n+1}^{L} (1-P_j^*) \right\} \qquad ,k=L-n \tag{37}$$

the decision rule is:

buy a second time if $q < q_k^*$

where $q_k^* = EH_k^* - (L + n)i$ (38)

Of course, the above derivation assumes that enough warehouse capacity is available which may not be true.

IV-C Conditions Required for Buying a Multiple of the Optimal Order Quantity

It has been shown in the previous section that there are conditions that make it advisable to purchase a quantity Q_o twice in the same inventory cycle. It is clear that if the cost of the item gets low enough before any purchase has been made in a cycle, it may be better to buy a quantity of $2Q_o$. There is one additional factor then which was not included in section IV-B. This factor is that if a purchase of $2Q_o$ is made then the cost of purchasing one order will be saved.

The different possibilities can be summarized as follows:

Case I A purchase of a quantity Q_o has already been made in the cycle.

Rule: Buy again a quantity Q_o, if in day k, $q \leq q_k^*$

Case II A purchase has not been made in the cycle yet.

Rules: Do not buy in day k if $q > q_k$

Buy quantity Q_o in day k if $q_k^o \leq q \leq q_k$

Buy quantity $2Q_o$ in day k if $q < q_k^o$

where $$q_k^o = q_k^* + \frac{A}{Q_o} \tag{39}$$

Notice that A is the ordering cost being saved and that A/Q_o are the savings per unit.

Assuming that enough warehouse capacity is available, price breaks similar to those calculated in section IV-B and here could be found for a third purchase, fourth purchase, etc.

IV-D Determination of Purchasing Decision Formulas when Backorders are Allowed

The deterministic backorder model presented here is the one given by Hadley and Whitin [8, page 42] modified to take advantage of price variations.

The symbols used are the following:

s - Number of backorders when a procurement arrives

π - Fixed backorder cost per unit

$\tilde{\pi}$ - Backorder cost per unit dependent on the length of time for which the backorder exists

λ - Demand per year

Q - Optimal order quantity

I - Inventory holding cost

C - Cost of the item

The behavior of the system is illustrated in Figure 16. Notice that after satisfying the backorders s upon the arrival of a procurement, the stock on hand is (Q-s). The time required for these (Q-s) units to be demanded is $T_1 = (Q-s)/\lambda$.

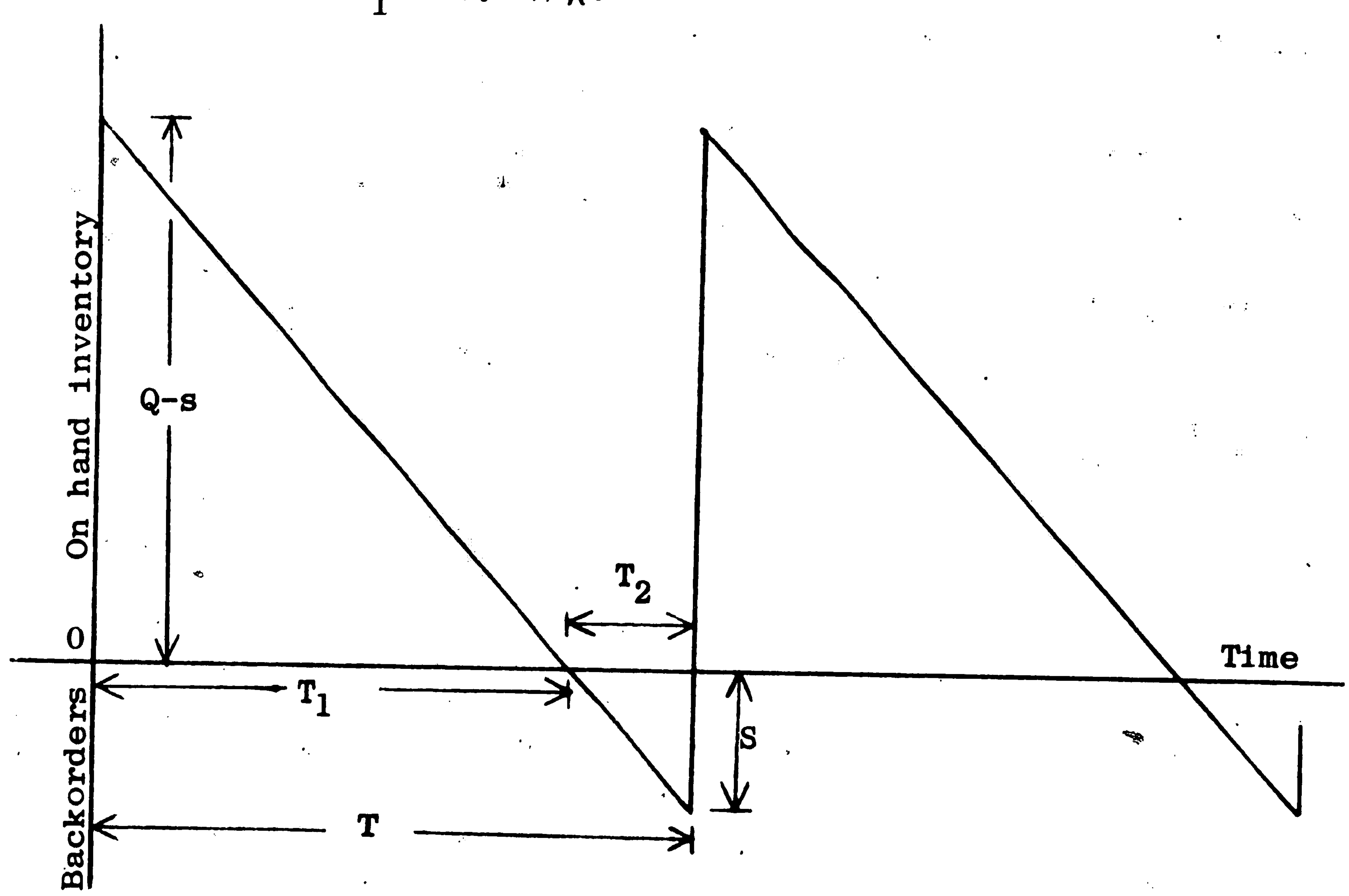


Figure 16 - Behavior of Backorder System

The inventory carrying costs per cycle are

$$IC\int_0^{T_1} (Q-s-\lambda t)dt = \frac{IC}{2\lambda}(Q-s)^2 \tag{40}$$

Since there are λ/Q cycles per year, then the average yearly cost of carrying inventory is

$$\frac{IC(Q-s)^2}{2Q} \tag{41}$$

The backorder cost per cycle is

$$\pi s + \tilde{\pi}\int_0^{T_2} \lambda t dt = \pi s + \tfrac{1}{2}\tilde{\pi}\lambda T_2^2$$

$$= \pi s + \frac{\tilde{\pi} s^2}{2} \quad (\text{since } T_2 = \frac{s}{\lambda}) \tag{42}$$

The average annual cost of backorders is

$$\frac{1}{Q}\left(\pi\lambda s + \tfrac{1}{2}\tilde{\pi} s^2\right) \tag{43}$$

The average annual variable cost K, which includes the cost of ordering, holding inventory, and backorders is then

$$K = \frac{\lambda}{Q} A + \frac{1}{2Q} IC (Q-s)^2 + \frac{1}{Q}\left[\pi\lambda s + \tfrac{1}{2}\tilde{\pi} s^2\right] \tag{44}$$

At this point Hadley and Whitin proceed to minimize equation (44) with respect to Q and s. It will be assumed here that this task has been successfully carried out and the necessary modifications to make use of unit price variations will be introduced in the next paragraph.

Before proceeding, the terminology used in the price breaks model and the one used by Hadley and Whitin in the backorders model have to be made consistent with each other. It is only necessary to keep in mind the following:

	Backorders Model		Price Break Model
Length of inventory cycle	T	=	L
Unit cost of the item	fixed		Variable
Average unit cost of the item	C	=	$\bar{q}$

The price break model for the backorder case is arrived at by effecting a cost trade-off as was done in Chapter II. The trade-off consists here of savings in the unit cost of the item and savings in the backorder penalty per item against extra charges in the inventory holding cost. Referring back to the basic price break formulas (3), (4), (7), (8), (13) in Chapter II it is easy to see that only formulas (4) and (13) need to be changed in this case. These are the formulas that give the expected unit price in a given period times the probability that a purchase is made in the period and the one that gives the price breaks respectively. The other formulas do not require any change since they are a function of the quantities given by (4) and (13).

There are two cases to consider. One of them is when the purchase is made before any backorder has been incurred, that is, when $n > T_2$ or $k < T_1$. The second case is when the purchase is made after some backorders exist, that is, when $n < T_2$ or $k > T_1$.* The first case will be studied now.

*The meaning of n,k was explained in Chapter II. The number of days passed from the beginning of the cycle is k and the number of days left in the cycle is n. So $n + k = L$.

If the purchase is made when $k < T_1$, all the backorders cost for this cycle are saved and the savings per unit are

$$\frac{1}{Q}\left[\pi s + \tilde{\pi}\, s\, T_2/2\right] \tag{45}$$

The additional inventory holding cost per unit which was given before by (L-k) i is now less than this amount. This is shown clearly in Figure 17.

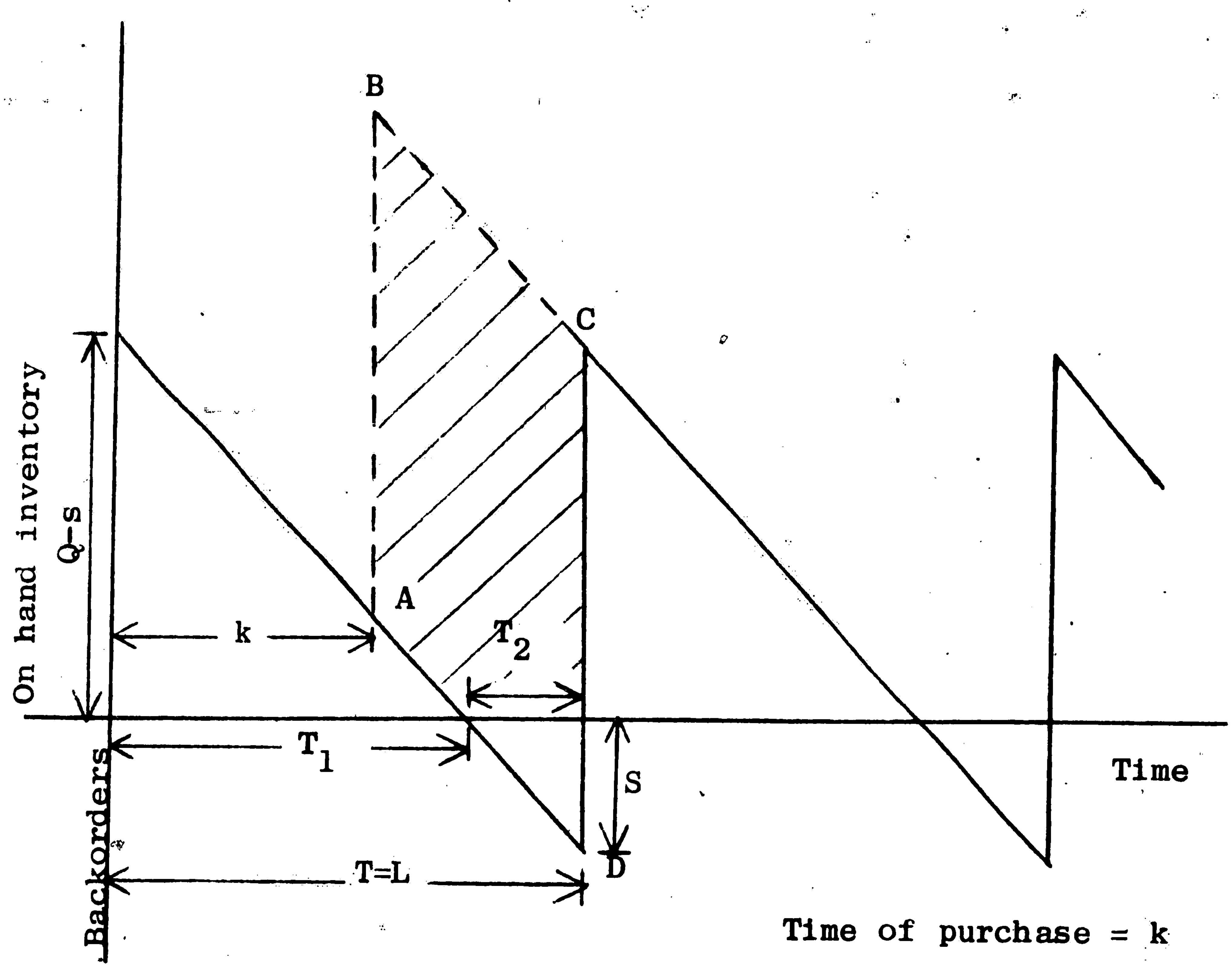


Figure 17 - Holding Cost in the Modified Backorder System

The holding costs when no backorders were allowed was proportional to the polygonal ABCD, while now it is proportional to the lined polygonal. The new additional holding costs per unit weighted by the appropriate factor is then

$$(L-k)i\left[\frac{(L-k)Q - T_2\, s/2}{(L-k)Q}\right]$$

or

$$i\left[(L-k) - s^2/2\lambda Q\right] \tag{46}$$

Now let:

$$M_k = i\left[(L-k) - s^2/2\lambda Q\right] - \frac{1}{Q}\left[\pi s + \tilde{\pi}\, s\, T_2/2\right] \tag{47}$$

then the formulas equivalent to (4) and (13) are respectively

$$e_k = \int_0^{q_k} (q + M_k)\, f(q)dq \tag{48}$$

$$q_k = EH_k - M_k \tag{49}$$

The second case, that is, when $k > T_1$ is more difficult to handle. The situation now is illustrated in Figure 18. This case will be studied in two parts. The variable backorders cost will be considered first.

The variable backorder cost per cycle was before $\tilde{\pi} s^2/2\lambda$. This quantity is proportional to the area of triangle EGD in Figure 18. The new variable backorder cost must be proportional to the area of the triangle EFA. The new backorder cost is then given by

$$\tilde{\pi}\, s\, \frac{(T_2 - n)}{T_2} \quad \frac{(T_2 - n)}{2} \tag{50}$$

and using $T_2 = s/\lambda$, the result is

$$\tilde{\pi}\, s\, \frac{(s/\lambda - n)}{s/\lambda} \quad \frac{(s/\lambda - n)}{2} = \frac{\tilde{\pi}}{2\lambda}\,(s - \lambda n)^2 \tag{51}$$

the savings per cycle are then

$$\frac{\tilde{\pi}\, s^2}{2\lambda} - \frac{\tilde{\pi}}{2\lambda}\,(s^2 - 2s\lambda n + \lambda^2 n^2) = \frac{\tilde{\pi}}{2}\,(2sn - \lambda n^2) \tag{52}$$

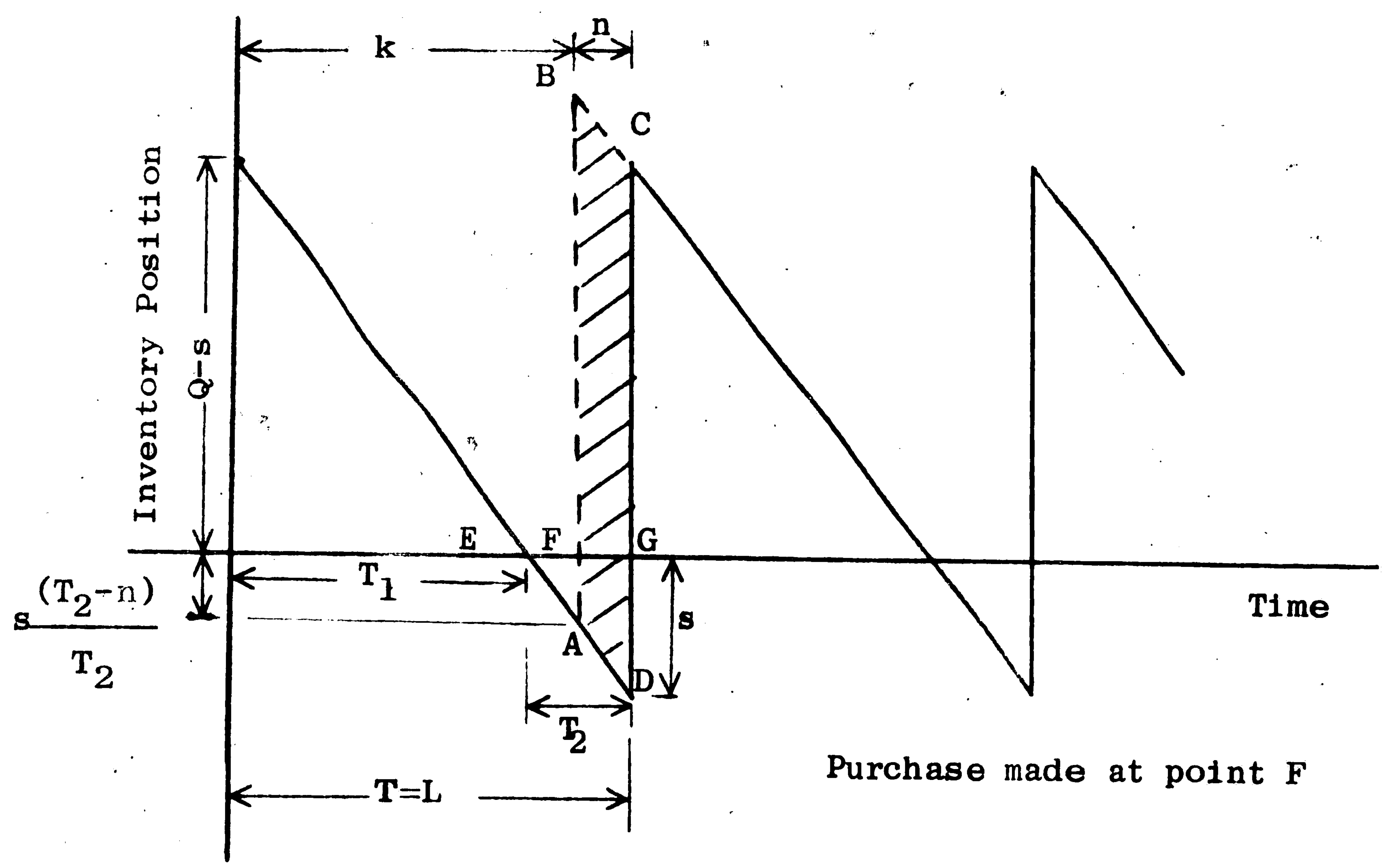


Figure 18. Modified Backorder Model - Second Case

This expression may be checked by evaluating it at the boundary points. For example when $n = T_2$, the savings should equal the total variable cost

$$\text{Savings } (n = T_2) = \frac{\tilde{\pi}}{2}\left(2s\,\frac{s}{\lambda} - \lambda\,\frac{s^2}{\lambda^2}\right) = \frac{\tilde{\pi}}{2}\,\frac{s^2}{\lambda}$$

Similarly a check can be made at the point $n = 0$ where the savings should be zero.

Now consider the fixed backorders cost. This cost is proportional to the negative ordinate in Figure 18 at the time the purchase is made. Initially it was πs, but now it will be

$$\pi s\,\frac{(T_2 - n)}{T_2} = \frac{\pi s\,(k-T_1)}{T_2} = \pi s\,\frac{s/\lambda - n}{s/\lambda} = \pi(s-\lambda n) \tag{53}$$

The savings in fixed backorder costs per cycle will be then

$$\pi s - \pi(s - \lambda n) = \pi\lambda n \tag{54}$$

Again a check can be made evaluating this quantity at the boundary points. If $n = T_2$, then the savings should be equal to the total fixed backorder cost. This is shown in (55)

$$\text{Savings } (n = T_2) = \pi\lambda T_2 = \pi\lambda\frac{s}{\lambda} = \pi s \tag{55}$$

Now only the holding cost remains to be considered. The new holding cost must be weighted by the ratio of the area of polygon BCGF to that of polygon BCDA in Figure 18. The new holding cost per unit per period is then

$$(L-k)i\left[\frac{(L-k)Q - (S(T_2-n)/T_2 + s)(L-k)/2}{(L-k)Q}\right] \tag{56}$$

This can be simplified to the following equation

$$(L-k)i \left[1 - \frac{1}{2Q}\ (2s - \lambda n)\right] \tag{57}$$

Before obtaining the new expressions for e_k and q_k, equations (52) and (54) must be put in a unit basis by dividing over Q since they were savings per cycle. The following equation for M_k is obtained combining equations (52), (54), and (57).

$$M_k = (L-k)i \left[1 - \frac{1}{2Q}\ (2s - \lambda n)\right] - \frac{1}{Q}\left[\frac{\tilde{\pi} n}{2}\ (2s - \lambda n) + \pi \lambda n\right] \tag{58}$$

and the formulas for e_k and q_k are the same as before:

$$e_k = \int_0^{q_k} (q + M_k)\ f(q)dq \tag{59}$$

$$q_k = EH_k - M_k \tag{60}$$

In conclusion then, it can be said that the only changes required are the use of formulas (59) and (60) in place of (4) and (13) keeping in mind the following conditions: For price breaks at a time $k \leq T_1$, use M_k as given by formula (47). If $k \geq T_1$, use M_k as given by formula (58).

V. MODELS FOR STOCHASTIC DEMAND CONDITIONS

There are two approaches that can be taken when both the demand and the price of the item are subject to stochastic variations. In one of them both parameters are considered simultaneously. The only method available in this case is dynamic programming and the results would necessarily be very complicated. Some limited attempts at following this approach have been reported in the literature as mentioned in Chapter 1. The other approach involves fixing the price of the item at its mean and optimizing the system with respect to the stochastic demand. After the system parameters have been found in this fashion, the variable price property is introduced and the system modified accordingly. This second approach, although a sub-optimization, is a lot simpler and more practical. Two heuristic models using the second approach are presented next.

V-A Development of a "Modified Q" System Model

The standard Q system has a fixed order size and a varying order period. It is completely determined by specifying the order quantity Q and the reorder point s. These quantities are calculated by balancing the ordering cost and out of stock cost against the inventory holding cost.

The price of the item itself is variable in the problem described here. This price may drop to a certain level, say q, when there are more than s units left such that it will be better to buy right away rather than wait for the reorder point. When a purchase is made before reaching the reorder point due to a low price quotation, money is saved on the unit cost and on stockout penalty costs. On the other hand there are extra expenses caused by additional inventory holding costs. The ordering cost does not change since the average number of orders per year remains the same.

The model discussed here is a modification of a model presented by Hadley and Whitin (8, Section 4-2). The assumptions made in the first part of the development are:

1. The unit cost C of the item is a constant.

2. The backorder cost is π per unit backordered.

3. There is a fixed lead time and there is never more than a single order outstanding.

4. The reorder point r is positive.

The total cost equation as given by Hadley and Whitin is

$$TC = \frac{\lambda A}{Q} + IC\left[\frac{Q}{2} + r - u\right] + \frac{\pi\lambda}{Q}\left[\int_r^{\infty} xh(x)dx - rH(r)\right] \tag{61}$$

The reader is referred to the reference if justification of any of the above terms is necessary. The meaning of the symbols is as follows:

TC - Total cost per year

λ - Demand in units per year

A - Ordering cost

Q - Lot size

I - Holding cost per dollar per year

C - Unit cost of the item

s - safety stock = (r-u)

r - Reorder point

u - Expected lead time demand = $\int_0^\infty xh(x)dx$

h(x) - lead time demand distribution if lead time is a constant

$= f(x;\bar{\tau})$

$H(r) - \int_r^\infty h(x)dx$

To obtain the optimum solution for the first part of the problem, equation (61) is minimized with respect to Q and s. The conventional problem has been described up to here. The use of the price variations will be described next.

Let ψ be the average number of units sold per time period. This time period can be hours, days, weeks, etc. and it fixes the interval at which price quotations should be checked. Again the only equations that need to be modified are the ones for e_k and q_k. The other equations stay the same as in Chapter II. The result is

$$e_k = \int_0^{q_k} (q + M_k)\, f(q)dq \tag{62}$$

$$q_k = EH_k - M_k \tag{63}$$

$$M_k = (L-k)i + \frac{\pi}{Q} \int_{r+\psi n}^{\infty} (x-r-\psi n)h(x)dx - \frac{\pi}{Q}\int_r^\infty (x-r)h(x)dx \tag{64}$$

Notice that M_k is simply a modification of the expression used before, to account for the savings in out of stock costs caused by ordering early.

V-B Development of a "Modified P" System Model

The family of P system models has a fixed order period, a variable order quantity, and a fixed inventory position after ordering. So in general the P systems are completely specified by the time between orders and the level for the inventory position immediately after placing an order.

To introduce the "price break" formulas in a P system, the model has to be considered in a cycle basis. Every time a purchase is made, a new cycle must be started. Using the price breaks, a purchase will be made in any period k for whatever quantity is necessary to bring the inventory level to the optimum position R if the price during that period drops below q_k.

The quantities that optimize the conventional P system are obtained by balancing the ordering costs, inventory holding costs and stockout penalty costs. The possibility of having variable costs is introduced here, as it has been in other models in this thesis. Again it is proposed to take advantage of the variable costs by buying before the order period has expired if the price is low enough. The quantities that have to be balanced in this second optimization of the system are the savings resulting from the cost of the items and in the stockout costs against the increased expenses of holding inventories and ordering.

The behavior of the system is illustrated in Figure 19. The solid line represents the conventional P system and the broken line gives the system being proposed. Notice that in the proposed system a new period starts every time that a purchase is made. Also notice that a purchase is made anywhere from time 0 to time P after the previous one.

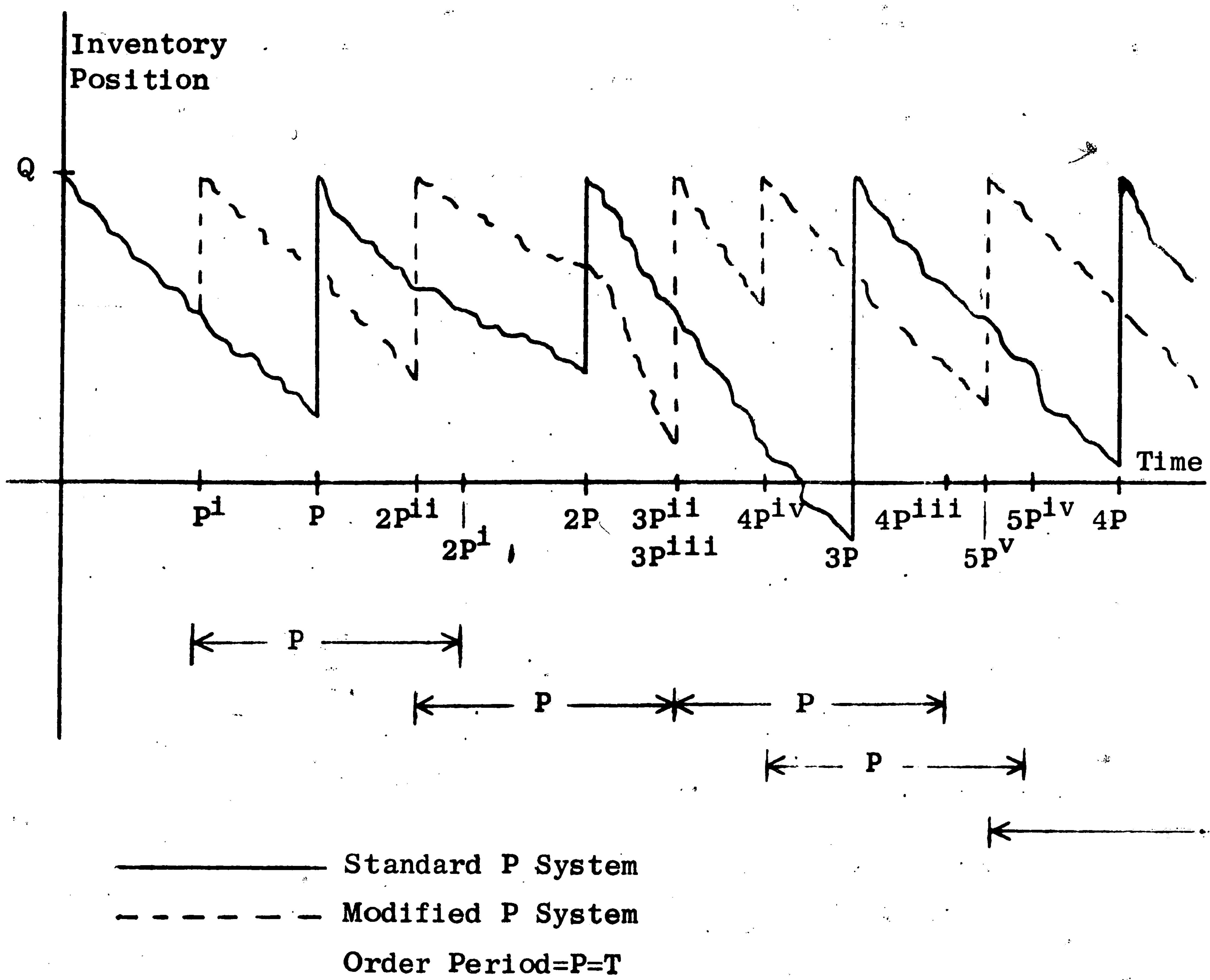


Figure 19. Modified P System

The P system model given below is the one described by Hadley and Whitin (8, pp. 237-241). The time between orders is denoted by T and at each review time a sufficient quantity is ordered to bring the inventory position of the system up to a level R(R=Q). The initial problem is to determine the optimal values for R and T given the total cost equation (65). The above step, the justification of each term in the equation and the assumptions made, are given by Hadley and Whitin. These matters are not discussed further here since the purpose of this paper is to discuss modifications that make use of the price variations after the parameters that optimize the conventional model have been found.

The total cost equation is

$$TC = \frac{B}{T} + IC\left[R-u-\frac{\lambda T}{2}\right] + \frac{\pi}{T}\int_{R}^{\infty}(X-R)f(X;\ \tau+T)dx \tag{65}$$

where

TC - total cost per year

B - ordering cost

T - time between reviews

I - inventory holding costs in percent/100 per unit per year

C - cost of the item

R - inventory position immediately after reviewing

u - expected lead time demand

λ - average demand rate in units/year

X - demand

π - backorder cost per unit backordered per period

$f(X;\tau+T)$ - frequency function of the demand furing lead time plus one order period

τ - procurement lead time

As in previous cases, the equations that have to be modified in the price break system are the ones for e_k and q_k. The new equations are

$$e_k = \int_0^{q_k} (q + M_k)\, f(q)dq \tag{66}$$

$$q_k = EH_k - M_k \tag{67}$$

where

$$M_k = (L-k)i - \frac{1}{\lambda T} \int_R^{\infty} (X-R) \left[f(X;\tau+T) - f(X;\tau+T-n) \right] - \frac{B}{\lambda} \left[\frac{1}{T} - \frac{1}{T-n} \right] \tag{68}$$

Notice that both T and n must be in the same units. The first term in equation (68) takes care of the additional holding costs, the second term considers the savings in stockout costs and the last one accounts for the higher ordering costs per year.

The modification made here for one P system model can be used as a guide to modify any other model. The use of the approach shown constitutes a trade-off between more general and sophisticated but complicated models on the one hand (if they were available) and simplicity at the price of suboptimization in the other hand.

VI. RECOMMENDATIONS FOR FUTURE STUDY

Several ideas and heuristic models have been presented in this paper. However, no attempt has been made to use any of them in practical cases. The most important task left regarding these models is to use them. This task involves finding the application, collecting data, estimating parameters, using the model and then testing the results.

Other areas that require additional investigation are the following:

1. Development of a model that does not assume independence between the price of the item in one period and the price in the following period.
2. Development of a price break model for an inflationary market in which the price trend is known. This is really a special case of #1.
3. Incorporation of the price breaks model to some dynamic models such as the Wagner-Whitin model [13, section 30]. The approach should be to use the dynamic algorithm to compute optimum purchase quantities at specified times using average costs. Then use the price break model for trade offs regarding the purchase time for each lot.
4. Possibility of using forecasted demands and prices together with the price breaks model. Some heuristic rules making use of forecasting have been proposed by Fabian and others [3].

VII. SUMMARY AND CONCLUSIONS

The purpose of this paper was to modify existing inventory models to take advantage of variations in the costs of the items. This objective has been realized.

The modified standard inventory model under deterministic demand conditions was studied in detail. To make use of the modified model, it is required that the user know or predict the probability distribution for the cost of the item in each time period. It has been shown, however, that the parameters requiring prediction are not very critical. The results obtained using the price break model, even if a mistake is made in the prediction of the form of the cost distribution or in the estimation of its mean, are considerably better than when the model was not used as has been shown in Chapter III. A method to calculate the amount of error required in the mean before the above statement ceases to be true was presented in the same chapter.

The savings realized by the model are directly proportional to the variance of the cost distribution. Of course, if the variance is zero, the cost is fixed and no savings are produced.

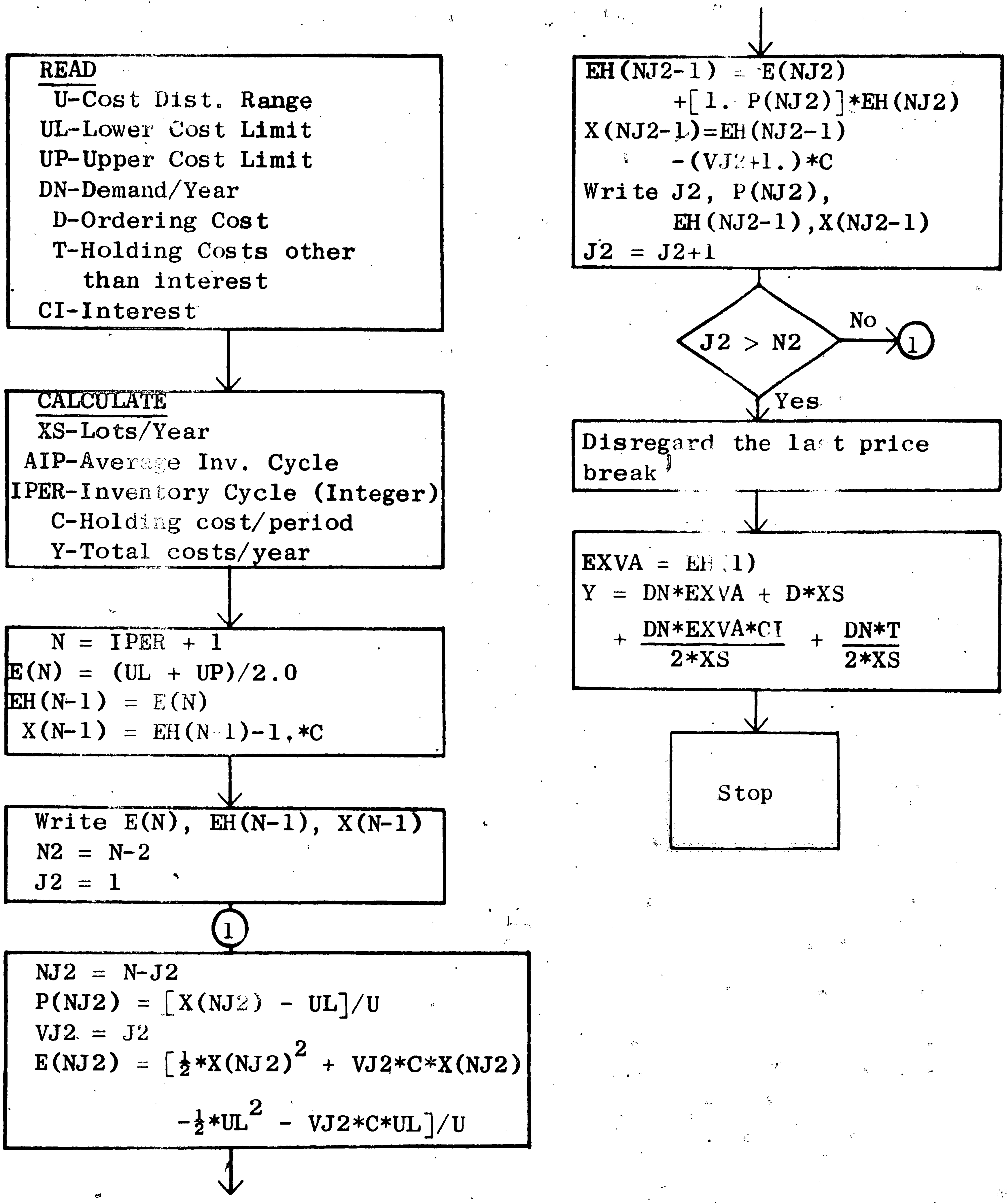
Several extensions of the basic model for the deterministic demand case were presented. They include the establishment of conditions required to purchase the optimum lot quantity more than once in an inventory cycle, conditions required to purchase a multiple of the optimal order quantity in a given inventory cycle and the development of a model that allows backorders.

Two heuristic models to make use of price variations in systems with stochastic demands were also presented. These models were obtained by modifying the standard Q system model (fixed order quantity) and p system model (fixed order period). The modification consisted of introducing a method to balance the savings and additional expenses caused by buying before it was required, to take advantage of low prices of the item.

APPENDIX I

Calculation of Price Breaks (Uniform Distribution)

A - Flow Chart for Program using Formula (9)
B - Computer Program using Formula (9)
C - Sample Output



Calculations of Price Breaks and Expected Unit Cost - Uniform Distribution of Costs

Flow Chart for Program using Formula (9)

Appendix 1-A

```
C     PRICE BREAKS FOR OPTIMUM PURCHASING POLICIES IN AN INVENTORY SYS.(UNIFORM)
      DIMENSION E(90),EH(90),X(90),P(90)
   24 FORMAT (6F10.0,F5.0)
   25 FORMAT(1H1,36HINVENTORY MODEL USING BAYES CRITERIA///)
   26 FORMAT(1H ,8HLOTS(XS),16X,4HCOST,8X,6HPERIOD,10X,12HDAILY CHARGE/)
   27 FORMAT(1H ,F10.4,4X,E14.8,4X,F10.4,4X,I4,4X,F10.5///)
    2 FORMAT (1H ,4X,2HJ2,12X,1HP,18X,1HE,17X,2HEH,18X,1HX//)
    3 FORMAT(1H ,5X,1H0,18X,E14.8, 5X,E14.8, 5X,E14.8/)
    4 FORMAT(1H ,3X,I3,3X,F10.7, 5X,E14.8, 5X,E14.8, 5X,E14.8/)
    5 FORMAT(1H ,30HDISREGARD THE LAST PRICE BREAK//)
    6 FORMAT(1H ,37HEXPECTED TOTAL COST WITH PRICE BREAKS,3X,E14.8)
   11 FORMAT(1H ,32HPRICE BREAKS AND EXPECTED VALUES///)
   99 FORMAT(' ')
   15 FORMAT(34HEXPECTED PRICE FOR PURCHASE PERIOD,E14.8,2X,I5,2X,F10.3)
   12 FORMAT(10HDAYS LEFT=,I3,7X,E14.8)
   23 READ(2,24)U,UL,UP,DN,D,T,CI
      WRITE(3,25)
      XS=((DN*CI*(UL+UP)/2.+DN*T)/(2.*D))**(0.5)
      AIP=365./XS
      IPER=AIP
      C=(CI*(UL+UP)/2.0+T)/365.0
      Y=DN*(UL+UP)/2.0+ D*XS+ (DN*(UL+UP)*CI)/(4.0*XS)+ (DN*T)/(2.0*XS)
      WRITE(3,26)
      WRITE(3,27)XS,Y,AIP,IPER,C
   50 WRITE(3,11)
      WRITE(3,2)
      N=IPER+1
      E(N)=(UL+UP)/2.0
      EH(N-1)=E(N)
      X(N-1)=EH(N-1)-1.*C
      WRITE(3,3) E(N),EH(N-1),X(N-1)
      N2=N-2
      DO 102 J2=1,N2
      NJ2=N-J2
      P(NJ2)=(X(NJ2)-UL)/U
      VJ2=J2
```

```
      E(NJ2)=((X(NJ2)*X(NJ2))/2.0+VJ2*C*X(NJ2)-(UL*UL)/2.0-VJ2*C*UL)/U
      EH(NJ2-1)=E(NJ2)+(1.-P(NJ2))*EH(NJ2)
      X(NJ2-1)=EH(NJ2-1)-(VJ2+1.0)*C
  102 WRITE(3,4) J2,P(NJ2),E(NJ2),EH(NJ2-1),X(NJ2-1)
      WRITE(3,5)
      EXVA=EH(1)
      Y=DN*EXVA +D*XS +(DN*EXVA*CI)/(2.0*XS) +(DN*T)/(2.0*XS)
      WRITE(3,6) Y
      WRITE(2,99)
      WRITE(2,15) EH(1),IPER,AIP
      N1=N-1
      DO 14 I=1,N1
      L=N-I
   14 WRITE(2,12) I,X(L)
      CALL EXIT
      END
```


Computer Program using Formula (9)

Appendix 1-B

INVENTORY MODEL USING BAYES CRITERIA

LOTS (XS)	COST	PERIOD		DAILY CHARGE
35.7421	0.77714837E 06	-10.2120	10	1.00000

PRICE BREAKS AND EXPECTED VALUES

J2	P	E	EH	X
0		0.11000002E 04	0.11000002E 04	0.10990002E 04
1	0.4950000	0.51999694E 03	0.10754968E 04	0.10734968E 04
2	0.3674829	0.38172125E 03	0.10619914E 04	0.10589914E 04
3	0.2949560	0.30454003E 03	0.10532905E 04	0.10492905E 04
4	0.2464514	0.25351065E 03	0.10472160E 04	0.10422160E 04
5	0.2110791	0.21658938E 03	0.10427600E 04	0.10367600E 04
6	0.1837988	0.18827938E 03	0.10393811E 04	0.10323811E 04
7	0.1619043	0.16565814E 03	0.10367587E 04	0.10287587E 04
8	0.1437927	0.14701001E 03	0.10346901E 04	0.10256901E 04
9	0.1284497	0.13125503E 03	0.10330397E 04	0.10230396E 04

DISREGARD THE LAST PRICE BREAK

EXPECTED TOTAL COST WITH PRICE BREAKS 0.73014487E 06

Sample Output

APPENDIX 1-C

APPENDIX 2

Simulation of Price Break Inventory System

A - Description of the Simulation
B - Flow Chart for Simulation with Uniform Distribution
C - Computer Program for Simulation with Uniform Distribution
D - Computer Program for Simulation with Normal Distribution
E - Computer Program for Simulation with Increasing Triangular Distribution

II-A Description of the Simulations

The price breaks used in the simulation programs were calculated assuming a uniform distribution of price. The program in Appendix I was used.

Many of the symbols in the program do not agree with the symbols used before due to computer requirements regarding fixed and floating point variables. The most important symbols are explained below.

M	= Number of years in simulation
VAR	= Argument for random number generation in the computer
U	= Difference between maximum and minimum price
UL	= Minimum price
UP	= Maximum price
DN	= Demand in number of units per year
D	= Ordering cost
T	= Inventory cost (other than interest) per unit per year
CI	= Percent annual interest/100
EXVA	= Expected cost for the item using price breaks and including inventory holding costs
AIP	= Average inventory cycle in days
IPER	= Average inventory cycle in integer number of days (AIP - fractional days)
Y	= Total cost per year
XS	= Number of lots purchased per year
YP	= Cost per cycle during simulation

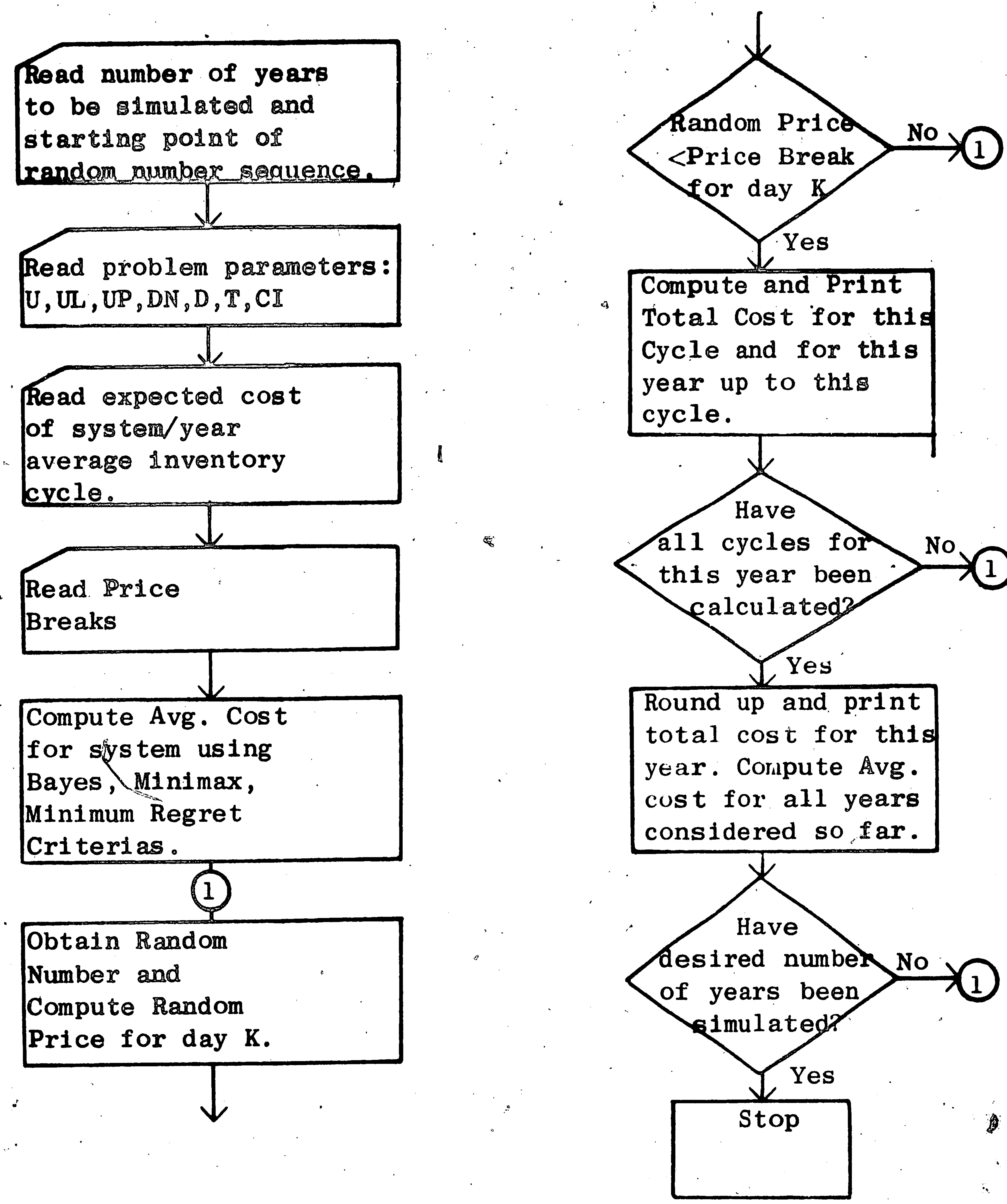
YRC = Cumulative cost during simulation (up to 1 year)

ATC = Average yearly cost

In the first program it is assumed that the actual price distribution is uniform. One input card is required stating the number of years to be simulated and the initial argument for the random number generation. Another gives the price limits, demand, ordering cost, inventory holding cost, and interest. The rest of the cards giving the expected cost of the item, length of the inventory cycle and the price breaks are obtained from the program in Appendix I.

The second program assumes that the actual price distribution is normal, but considers the minimum and maximum prices to correspond to the $\pm 2\ \sigma$ points. During the random quotation of prices any value outside the $\pm 2\ \sigma$ limit is considered equal to the maximum or minimum price as required. The inputs required are the same as the ones used in the first program.

The third program assumes that the probability function of price has triangular shape with 0 probability for the lower price limit and highest probability for the maximum price. This is then an increasing triangular distribution. A Monte Carlo simulation which requires feeding a cumulative distribution curve to the computer was not used in this case. Instead, a simple mathematical conversion formula which changes a random uniform number to a random triangular number was used. The formula is $b = U\sqrt{R}$ where b is the random number with triangular distribution, U is the size of the base of the triangle and R is a random uniform number.



II-B Flow Chart for Simulation with Uniform Distribution

Appendix II-B

```
C     CALCULATIONS AND SIMULATIONS-RECTANGULAR DISTRIBUTION
      DIMENSION X(100)
    5 FORMAT(I3,F5.5)
   10 FORMAT(F10.0,F10.0,F10.0,F10.0,F10.0,F10.0,F5.0)
   15 FORMAT(34X,E14.8,2X,I5,2X,F10.3)
   25 FORMAT(20X,E14.8)
   30 FORMAT(1H1,35HBAYES CRITERIA WITHOUT PRICE BREAKS/)
   35 FORMAT(1H ,11HNO. OF LOTS,2X,E14.8,5X,11HANNUAL COST,2X,E14.8///)
   40 FORMAT(1H ,41HEXPECTED BAYES CRITERIA WITH PRICE BREAKS/)
   45 FORMAT(1H ,16HMINIMAX CRITERIA/)
   50 FORMAT(1H ,23HMINIMUM REGRET CRITERIA/)
   55 FORMAT(///1H ,24HRANDOM NUMBER SIMULATION//)
   80 FORMAT(//1H ,4HYEAR,I4,15HPURCHASE PERIOD,I4)
   85 FORMAT(/1H ,12HPURCHASE DAY,I4,2X,5HPRICE,E14.8,2X,4HCOST,E14.8//)
   95 FORMAT(1H ,19HTOTAL COST FOR YEAR,I4,2X,2HIS,2X,E14.8/)
   96 FORMAT(1H ,18HAVERAGE COST FOR  ,I3,5HYEARS,4X,E14.8///)
  105 FORMAT(1H ,19HAVERAGE COST DURING,I4,2X,8HYEARS IS,2X,E14.8)
      READ(2,5) M,VAR
      READ(2,10) U,UL,UP,DN,D,T,CI
      READ(2,15) EXVA,IPER,AIP
      N=IPER-1
      DO 20 I=1,N
   20 READ(2,25) X(I)
      XS=((DN*CI*(UL+UP)/2.+DN*T)/(2.*D))**(0.5)
      XSB=XS
      Y=DN*(UL+UP)/2.0+ D*XS+ (DN*(UL+UP)*CI)/(4.0*XS)+ (DN*T)/(2.0*XS)
      WRITE(3,30)
      WRITE(3,35) XS,Y
      Y=DN*EXVA +D*XS +(DN*EXVA*CI)/(2.0*XS) +(DN*T)/(2.0*XS)
      WRITE(3,40)
      WRITE(3,35) XS,Y
      XS=(DN*(UP*CI+T)/(2.0*D))**(0.5)
      Y=DN*(UL+UP)/2.0+ D*XS+ (DN*(UL+UP)*CI)/(4.0*XS)+ (DN*T)/(2.0*XS)
      WRITE(3,45)
      WRITE(3,35) XS,Y
      XS2=XS
```

```
      XS1=(DN*(UL*CI+T)/(2.0*D))**(0.5)
      ANUM=DN*CI*(UP-UL)
      DEN=2.*D*(XS2-XS1)+(2.*D*DN*(UP*CI+T))**.5-(2.*D*DN*(UL*CI+T))**.5
      XS=ANUM/DEN
      Y=DN*(UL+UP)/2.0+ D*XS+ (DN*(UL+UP)*CI)/(4.0*XS)+ (DN*T)/(2.0*XS)
      WRITE(3,50)
      WRITE(3,35) XS,Y
      WRITE(3,55)
      MXS=XSB
      XSBR=MXS
      STC=0.0
      DO 100 L=1,M
      YPC=0.0
      DO 90 K=1,MXS
      DO 70 I=1,N
      VAR=VAR*83.
      NVAR=VAR
      VAR1=NVAR
      VAR=VAR-VAR1
      R=VAR
      RP=UL+R*U
      J=IPER-I
      IF(RP-X(J)) 75,75,65
   65 CONTINUE
   70 CONTINUE
      VAR=VAR*83.
      NVAR=VAR
      VAR1=NVAR
      VAR=VAR-VAR1
      R=VAR
      RP=UL+R*U
      J=IPER-I
   75 CONTINUE
      AEL=J
      AL=AEL+AIP
      YP=DN*RP/XSB+D+(DN*RP*CI*AL)/(2.*XSB*365.)+(DN*T*AL)/(2.*XSB*365.)
```

```
      YPC=YPC+YP
      WRITE(3,80) L,K
      WRITE(3,85) I,RP,YP
   90 CONTINUE
      TC=YPC*XSB/XSBR
      WRITE(3,95) L,TC
      STC=STC+TC
      VL=L
      ATC=STC/VL
      WRITE(3,96) L,ATC
  100 CONTINUE
      WRITE(3,105) M,ATC
      CALL EXIT
      END
```

Computer Program for Simulation with Uniform Distribution

Appendix 2-C

```
C     SIMULATION WITH NON UNIFORM DISTRIBUTION (NORMAL)
      DIMENSION X(100)
    5 FORMAT(I3,F5.5)
   10 FORMAT(F10.0,F10.0,F10.0,F10.0,F10.0,F10.0,F5.0)
   15 FORMAT(34X,E14.8,2X,I5,2X,F10.3)
   25 FORMAT(20X,E14.8)
   30 FORMAT(1H1,35HBAYES CRITERIA WITHOUT PRICE BREAKS/)
   35 FORMAT(1H ,11HNO. OF LOTS,2X,E14.8,5X,11HANNUAL COST,2X,E14.8///)
   45 FORMAT(1H ,16HMINIMAX CRITERIA/)
   50 FORMAT(1H ,23HMINIMUM REGRET CRITERIA/)
   55 FORMAT(///1H ,33HNORMAL RANDOM DEVIATES SIMULATION//)
   80 FORMAT(//1H ,4HYEAR,I4,15HPURCHASE PERIOD,I4)
   85 FORMAT(/1H ,12HPURCHASE DAY,I4,2X,5HPRICE,E14.8,2X,4HCOST,E14.8//)
   95 FORMAT(1H ,19HTOTAL COST FOR YEAR,I4,2X,2HIS,2X,E14.8/)
   96 FORMAT(1H ,18HAVERAGE COST FOR  ,I3,5HYEARS,4X,E14.8///)
  105 FORMAT(1H ,19HAVERAGE COST DURING,I4,2X,8HYEARS IS,2X,E14.8)
      READ(2,5) M,VAR
      READ(2,10) U,UL,UP,DN,D,T,CI
      READ(2,15) EXVA,IPER,AIP
      N=IPER-1
      DO 20 I=1,N
   20 READ(2,25) X(I)
      XS=((DN*CI*(UL+UP)/2.+DN*T)/(2.*D))**(0.5)
      XSB=XS
      Y=DN*(UL+UP)/2.0+ D*XS+ (DN*(UL+UP)*CI)/(4.0*XS)+ (DN*T)/(2.0*XS)
      WRITE(3,30)
      WRITE(3,35) XS,Y
      XS=(DN*(UP*CI+T)/(2.0*D))**(0.5)
      Y=DN*(UL+UP)/2.0+ D*XS+ (DN*(UL+UP)*CI)/(4.0*XS)+ (DN*T)/(2.0*XS)
      WRITE(3,45)
      WRITE(3,35) XS,Y
      XS2=XS
      XS1=(DN*(UL*CI+T)/(2.0*D))**(0.5)
      ANUM=DN*CI*(UP-UL)
      DEN=2.*D*(XS2-XS1)+(2.*D*DN*(UP*CI+T))**.5-(2.*D*DN*(UL*CI+T))**.5
      XS=ANUM/DEN
```

```
      Y=DN*(UL+UP)/2.0+ D*XS+ (DN*(UL+UP)*CI)/(4.0*XS)+ (DN*T)/(2.0*XS)
      WRITE(3,50)
      WRITE(3,35) XS,Y
      WRITE(3,55)
      MXS=XSB
      XSBR=MXS
      STC=0.0
      DO 100 L=1,M
      YPC=0.0
      DO 90 K=1,MXS
      DO 70 I=1,N
      RNDEV=0.
      DO 21 JR=1,12
      VAR=VAR*83.
      NVAR=VAR
      VAR1=NVAR
      VAR=VAR-VAR1
   21 RNDEV=RNDEV+VAR
      R=RNDEV-6.
      IF (R) 61,65,65
   61 IF (R+2.) 62,63,63
   62 RP=UL
      GO TO 64
   63 RP=(UL+UP)/2. +R*U/4.
   64 J=IPER-I
      IF(RP-X(J)) 78,78,65
   65 CONTINUE
   70 CONTINUE
      RNDEV=0.
      DO 22 JR=1,12
      VAR=VAR*83.
      NVAR=VAR
      VAR1=NVAR
      VAR=VAR-VAR1
   22 RNDEV=RNDEV+VAR
      R=RNDEV-6.
```

```
      IF(R) 71,75,73
   71 IF(R+2.) 72,75,75
   72 RP=UL
      GO TO 76
   73 IF(R-2.) 75,75,74
   74 RP=UP
      GO TO 76
   75 RP=(UL+UP)/2. +R*U/4.
   76 J=IPER-I
   78 CONTINUE
      AEL=J
      AL=AEL+AIP
      YP=DN*RP/XSB+D+(DN*RP*CI*AL)/(2.*XSB*365.)+(DN*T*AL)/(2.*XSB*365.)
      YPC=YPC+YP
      WRITE(3,80) L,K
      WRITE(3,85) I,RP,YP
   90 CONTINUE
      TC=YPC*XSB/XSBR
      WRITE(3,95) L,TC
      STC=STC+TC
      VL=L
      ATC=STC/VL
      WRITE(3,96) L,ATC
  100 CONTINUE
      WRITE(3,105) M,ATC
      CALL EXIT
      END
```

Computer Program for Simulation with Normal Distribution

Appendix 2-D

```
C      SIMULATION WITH INCREASING TRIANGULAR DISTRIBUTION
       DIMENSION X(100)
    5  FORMAT(I3,F5.5)
   10  FORMAT(F10.0,F10.0,F10.0,F10.0,F10.0,F10.0,F5.0)
   15  FORMAT(34X,E14.8,2X,I5,2X,F10.3)
   25  FORMAT(20X,E14.8)
   30  FORMAT(1H1,35HBAYES CRITERIA WITHOUT PRICE BREAKS/)
   35  FORMAT(1H ,11HNO. OF LOTS,2X,E14.8,5X,11HANNUAL COST,2X,E14.8///)
   45  FORMAT(1H ,16HMINIMAX CRITERIA/)
   50  FORMAT(1H ,23HMINIMUM REGRET CRITERIA/)
   55  FORMAT(///1H ,38HINCREASING TRIANGULAR DIST. SIMULATION//)
   80  FORMAT(//1H ,4HYEAR,I4,15HPURCHASE PERIOD,I4)
   85  FORMAT(/1H ,12HPURCHASE DAY,I4,2X,5HPRICE,E14.8,2X,4HCOST,E14.8//)
   95  FORMAT(1H ,19HTOTAL COST FOR YEAR,I4,2X,2HIS,2X,E14.8/)
   96  FORMAT(1H ,18HAVERAGE COST FOR  ,I3,5HYEARS,4X,E14.8///)
  105  FORMAT(1H ,19HAVERAGE COST DURING,I4,2X,8HYEARS IS,2X,E14.8)
       READ(2,5) M,VAR
       READ(2,10) U,UL,UP,DN,D,T,CI
       READ(2,15) EXVA,IPER,AIP
       N=IPER-1
       DO 20 I=1,N
   20  READ(2,25) X(I)
       PINC=U/1.414
   26  PR=UL+PINC
       XS=((DN*CI*(UL+UP)/2.+DN*T)/(2.*D))**(0.5)
       XSB=XS
       Y=DN*PR +D*XS +(DN*PR*CI)/(2.0*XS) +(DN*T)/(2.0*XS)
       WRITE(3,30)
       WRITE(3,35) XS,Y
       XS=(DN*(UP*CI+T)/(2.0*D))**(0.5)
       Y=DN*PR +D*XS +(DN*PR*CI)/(2.0*XS) +(DN*T)/(2.0*XS)
       WRITE(3,45)
       WRITE(3,35) XS,Y
       XS2=XS
       XS1=(DN*(UL*CI+T)/(2.0*D))**(0.5)
       ANUM=DN*CI*(UP-UL)
```

```
      DEN=2.*D*(XS2-XS1)+(2.*D*DN*(UP*CI+T))**.5-(2.*D*DN*(UL*CI+T))**.5
      XS=ANUM/DEN
      Y=DN*PR +D*XS +(DN*PR*CI)/(2.0*XS) +(DN*T)/(2.0*XS)
      WRITE(3,50)
      WRITE(3,35) XS,Y
      WRITE(3,55)
      MXS=XSB
      XSBR=MXS
      STC=0.0
      DO 100 L=1,M
      YPC=0.0
      DO 90 K=1,MXS
      DO 70 I=1,N
      VAR=VAR*83.
      NVAR=VAR
      VAR1=NVAR
      VAR=VAR-VAR1
      R=VAR
      PRO1=U*(R**0.5)
   56 RP=UL+PRO1
      J=IPER-I
      IF(RP-X(J)) 75,75,65
   65 CONTINUE
   70 CONTINUE
      VAR=VAR*83.
      NVAR=VAR
      VAR1=NVAR
      VAR=VAR-VAR1
      R=VAR
      PRO1=U*(R**0.5)
   71 RP=UL+PRO1
      J=IPER-I
   75 CONTINUE
      AEL=J
      AL=AEL+AIP
      YP=DN*RP/XSB+D+(DN*RP*CI*AL)/(2.*XSB*365.)+(DN*T*AL)/(2.*XSB*365.)
```

```
      YPC=YPC+YP
      WRITE(3,80) L,K
      WRITE(3,85) I,RP,YP
   90 CONTINUE
      TC=YPC*XSB/XSBR
      WRITE(3,95) L,TC
      STC=STC+TC
      VL=L
      ATC=STC/VL
      WRITE(3,96) L,ATC
  100 CONTINUE
      WRITE(3,105) M,ATC
      CALL EXIT
      END
```

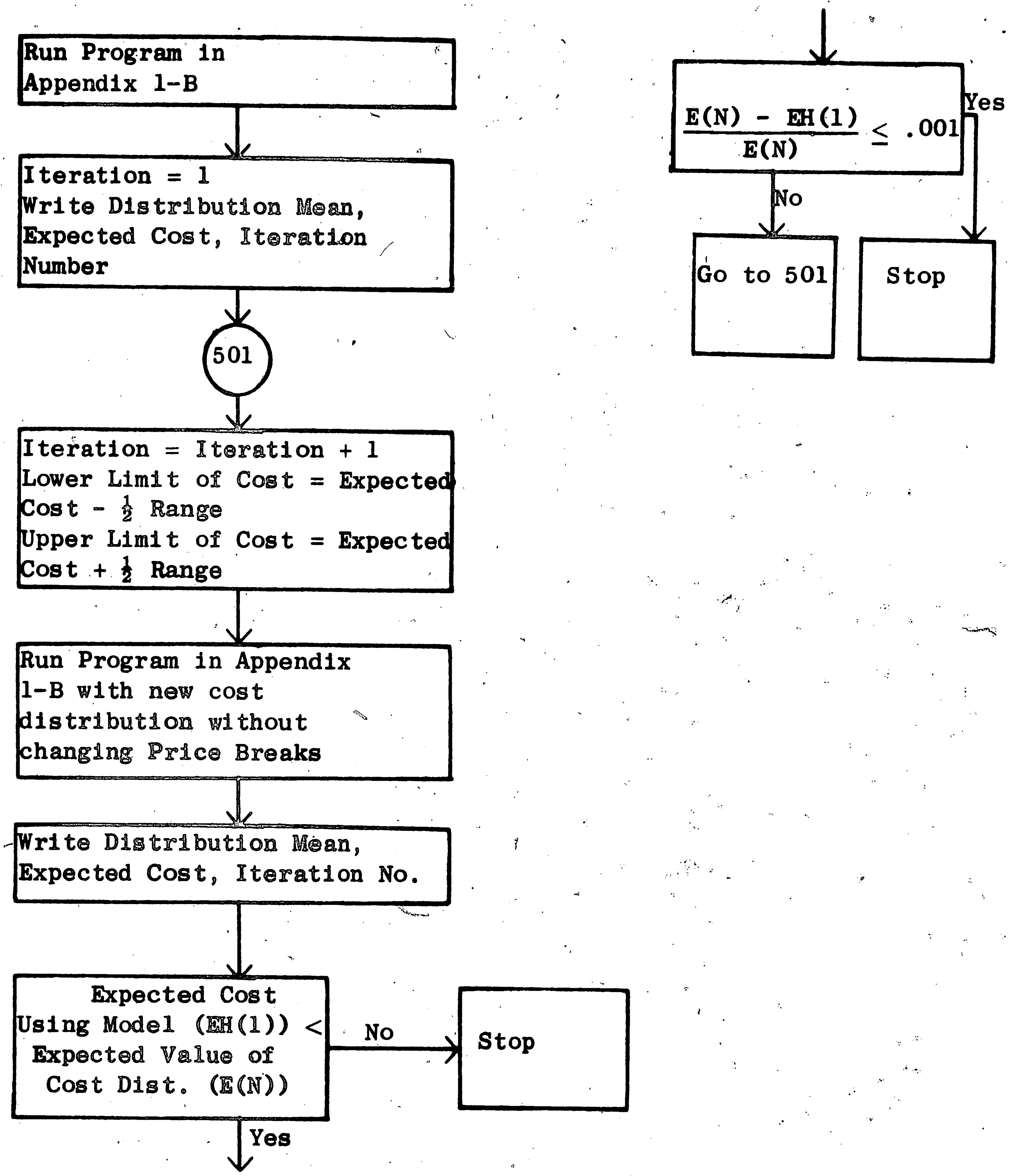
Computer Program for Simulation with Increasing Triangular Distribution

Appendix 2-E

APPENDIX 3

Determination of Confidence Limits for the Use of the Model

A - Flow Chart for Computer Program Assuming Uniform Distribution
B - Computer Program
C - Sample Output



Determination of Confidence Limits for the Use of the Model - Uniform Distribution

Flow Chart for Computer Program Assuming Uniform Distribution

Appendix 3-A

```
C      DETERMINATION OF CONFIDENCE LIMITS FOR THE ESTIMATED MEAN
       DIMENSION E(90),EH(90),X(90),P(90)
    24 FORMAT (6F10.0,F5.0)
    25 FORMAT(1H1,36HINVENTORY MODEL USING BAYES CRITERIA///)
    26 FORMAT(1H ,8HLOTS(XS),16X,4HCOST,8X,6HPERIOD,10X,12HDAILY CHARGE/)
    27 FORMAT(1H ,F10.4,4X,E14.8,4X,F10.4,4X,I4,4X,F10.5///)
     2 FORMAT (1H ,4X,2HJ2,12X,1HP,18X,1HE,17X,2HEH,18X,1HX//)
     3 FORMAT(1H ,5X,1H0,18X,E14.8, 5X,E14.8, 5X,E14.8/)
     4 FORMAT(1H ,3X,I3,3X,F10.7, 5X,E14.8, 5X,E14.8, 5X,E14.8/)
     5 FORMAT(1H ,30HDISREGARD THE LAST PRICE BREAK////)
    11 FORMAT(1H ,32HPRICE BREAKS AND EXPECTED VALUES///)
    31 FORMAT(1H ,18HITERATION NUMBER   ,I3/)
    32 FORMAT(1H ,23HPRICE DISTRIBUTION MEAN  ,E14.8/)
    33 FORMAT(1H ,16HEXPECTED PRICE  ,E14.8//)
    23 READ(2,24)U,UL,UP,DN,D,T,CI
       WRITE(3,25)
       XS=((DN*CI*(UL+UP)/2.+DN*T)/(2.*D))**(0.5)
       AIP=365./XS
       IPER=AIP
       C=(CI*(UL+UP)/2.0+T)/365.0
       Y=DN*(UL+UP)/2.0+ D*XS+ (DN*(UL+UP)*CI)/(4.0*XS)+ (DN*T)/(2.0*XS)
       WRITE(3,26)
       WRITE(3,27)XS,Y,AIP,IPER,C
    50 WRITE(3,11)
       WRITE(3,2)
       N=IPER+1
       E(N)=(UL+UP)/2.0
       EH(N-1)=E(N)
       X(N-1)=EH(N-1)-1.*C
       WRITE(3,3) E(N),EH(N-1),X(N-1)
       N2=N-2
       DO 102 J2=1,N2
       NJ2=N-J2
       P(NJ2)=(X(NJ2)-UL)/U
       VJ2=J2
       E(NJ2)=((X(NJ2)*X(NJ2))/2.0+VJ2*C*X(NJ2)-(UL*UL)/2.0-VJ2*C*UL)/U
```

```
      EH(NJ2-1)=E(NJ2)+(1.-P(NJ2))*EH(NJ2)
      X(NJ2-1)=EH(NJ2-1)-(VJ2+1.0)*C
  102 WRITE(3,4) J2,P(NJ2),E(NJ2),EH(NJ2-1),X(NJ2-1)
      WRITE(3,5)
      ITER=1
      WRITE(3,31) ITER
      WRITE(3,32) E(N)
      WRITE(3,33) EH(1)
   34 ITER=ITER+1
      UL=EH(1)-U/2.0
      UP=EH(1)+U/2.0
      E(N)=(UL+UP)/2.0
      EH(N-1)=E(N)
      N2=N-2
      DO 103 J2=1,N2
      NJ2=N-J2
      P(NJ2)=(X(NJ2)-UL)/U
      VJ2=J2
      E(NJ2)=((X(NJ2)*X(NJ2))/2.0+VJ2*C*X(NJ2)-(UL*UL)/2.0-VJ2*C*UL)/U
      EH(NJ2-1)=E(NJ2)+(1.-P(NJ2))*EH(NJ2)
  103 CONTINUE
      WRITE(3,31) ITER
      WRITE(3,32) E(N)
      WRITE(3,33) EH(1)
      IF(EH(1)-E(N))35,36,36
   35 A = (E(N)-EH(1))/E(N)
      IF (A-.001)36,36,34
   36 CALL EXIT
      END
```

Computer Program

Appendix 3-B

INVENTORY MODEL USING BAYES CRITERIA

LOTS(XS)	COST	PERIOD		DAILY CHARGE
35.7421	0.77714837E 06	10.2120	10	1.00000

PRICE BREAKS AND EXPECTED VALUES

J2	P	E	EH	X
0		0.11000002E 04	0.11000002E 04	0.10990002E 04
1	0.4950000	0.51999694E 03	0.10754970E 04	0.10734970E 04
2	0.3674841	0.38172253E 03	0.10619914E 04	0.10589914E 04
3	0.2949560	0.30454003E 03	0.10532905E 04	0.10492905E 04
4	0.2464514	0.25351065E 03	0.10472160E 04	0.10422160E 04
5	0.2110791	0.21658938E 03	0.10427600E 04	0.10367600E 04
6	0.1837988	0.18827938E 03	0.10393813E 04	0.10323813E 04
7	0.1619055	0.16565939E 03	0.10367590E 04	0.10287590E 04
8	0.1437939	0.14701126E 03	0.10346904E 04	0.10256904E 04
9	0.1284509	0.13125625E 03	0.10330395E 04	0.10230394E 04

DISREGARD THE LAST PRICE BREAK

ITERATION NUMBER 1

PRICE DISTRIBUTION MEAN0.11000002E 04

EXPECTED PRICE 0.10330395E 04

ITERATION NUMBER 2

PRICE DISTRIBUTION MEAN0.10330395E 04

EXPECTED PRICE 0.98933984E 03

ITERATION NUMBER 3

PRICE DISTRIBUTION MEAN0.98933972E 03

EXPECTED PRICE 0.96681323E 03

ITERATION NUMBER 4

PRICE DISTRIBUTION MEAN0.96681311E 03

EXPECTED PRICE 0.95540246E 03

ITERATION NUMBER 5

PRICE DISTRIBUTION MEAN0.95540246E 03

EXPECTED PRICE 0.94964392E 03

ITERATION NUMBER 6

PRICE DISTRIBUTION MEAN0.94964392E 03

EXPECTED PRICE 0.94674170E 03

ITERATION NUMBER 7

PRICE DISTRIBUTION MEAN0.94674157E 03

EXPECTED PRICE 0.94527990E 03

ITERATION NUMBER 8

PRICE DISTRIBUTION MEAN0.94527990E 03 $945.28 CRITICAL VALUE

EXPECTED PRICE 0.94454382E 03 $944.54

.74

Sample Output

Appendix 3-C

APPENDIX 4

Deviation from Optimality

A - Error in the Estimated Mean Cost - Computer Program (Uniform Distribution)

B - Error in the Price Breaks Used - Computer Program (Uniform Distribution)

```
C     SENSITIVITY ANALYSIS  2  -VARIATIONS IN THE MEAN OF A UNIFORM DIST
      DIMENSION E(90),EH(90),X(90),P(90),XV(90),PERC(51)
   21 FORMAT(I2)
    1 FORMAT(14F5.0)
   24 FORMAT (6F10.0,F5.0)
   25 FORMAT(1H1,36HINVENTORY MODEL USING BAYES CRITERIA//)
   26 FORMAT(1H ,8HLOTS(XS),16X,4HCOST,8X,6HPERIOD,10X,12HDAILY CHARGE/)
   27 FORMAT(1H ,F10.4,4X,E14.8,4X,F10.4,4X,I4,4X,F10.5////)
    2 FORMAT (1H ,4X,2HJ2,12X,1HP,18X,1HE,17X,2HEH,18X,1HX//)
    3 FORMAT(1H ,5X,1HO,18X,E14.8, 5X,E14.8, 5X,E14.8/)
    4 FORMAT(1H ,3X,I3,3X,F10.7, 5X,E14.8, 5X,E14.8, 5X,E14.8/)
    5 FORMAT(1H ,30HDISREGARD THE LAST PRICE BREAK//)
    6 FORMAT(' EXPECTED UNIT COST WITH OPT. PRICE BREAKS',3X,E14.8//)
    7 FORMAT(' EXPECTED UNIT COST W/O PRICE BREAKS',3X,E14.8//)
    8 FORMAT(' EXPECTED UNIT COST WITH INITIAL PRICE BREAKS',3X,E14.8//)
   11 FORMAT(/' PRICE BREAKS,EXP VAL-NEW MEAN=MEAN+',F5.2,'*STD DEV'//)
C     CALCULATION OF PRICE BREAKS AND EXPECTED ITEM COST FOR THE INITIAL
C     MEAN
      READ(2,21)NP
C     NP=NUMBER OF POINTS DESIRED
      READ(2,1) (PERC(I),I=1,NP)
      READ(2,24)U,UL,UP,DN,D,T,CI
      WRITE(3,25)
      XS=((DN*CI*(UL+UP)/2.+DN*T)/(2.*D))**(0.5)
      AIP=365./XS
      IPER=AIP
      C=(CI*(UL+UP)/2.0+T)/365.0
      Y=DN*(UL+UP)/2.0+ D*XS+ (DN*(UL+UP)*CI)/(4.0*XS)+ (DN*T)/(2.0*XS)
      WRITE(3,26)
      WRITE(3,27)XS,Y,AIP,IPER,C
      I=1
      WRITE(3,11)PERC(I)
      WRITE(3,2)
      N=IPER+1
      E(N)=(UL+UP)/2.0
      AVG=E(N)
```

```
      EH(N-1)=E(N)
      X(N-1)=EH(N-1)-1.*C
      WRITE(3,3) E(N),EH(N-1),X(N-1)
      N2=N-2
      DO 102 J2=1,N2
      NJ2=N-J2
      P(NJ2)=(X(NJ2)-UL)/U
      VJ2=J2
      E(NJ2)=((X(NJ2)*X(NJ2))/2.0+VJ2*C*X(NJ2)-(UL*UL)/2.0-VJ2*C*UL)/U
      EH(NJ2-1)=E(NJ2)+(1.-P(NJ2))*EH(NJ2)
      X(NJ2-1)=EH(NJ2-1)-(VJ2+1.0)*C
  102 WRITE(3,4) J2,P(NJ2),E(NJ2),EH(NJ2-1),X(NJ2-1)
      WRITE(3,5)
      WRITE(3,6)EH(1)
      WRITE(3,7)E(N)
C     CALCULATION OF VARIANCE AND STANDARD DEVIATION
      VAR=(U*U)/12.
      STDEV=VAR**(0.5)
      DO 31  I=2,NP
      WRITE(3,11) PERC(I)
      WRITE(3,2)
C     NEW MEAN=INITIAL MEAN +OR- SOME   OF THE STANDARD DEVIATION
      AMEAN=AVG+PERC(I)*STDEV
      UL=AMEAN-U/2.
      UP=AMEAN+U/2.
C     CALCULATION OF PRICE BREAKS AND EXPECTED ITEM COST USING THE NEW
C     MEAN
      E(N)=(UL+UP)/2.0
      EH(N-1)=E(N)
      XV(N-1)=EH(N-1)-1.*C
      WRITE(3,3) E(N),EH(N-1),XV(N-1)
      N2=N-2
      DO 103 J2=1,N2
      NJ2=N-J2
      P(NJ2)=(XV(NJ2)-UL)/U
      VJ2=J2
```

```
      E(NJ2)=((XV(NJ2)**2.)/2. +VJ2*C*XV(NJ2)-(UL*UL)/2.-VJ2*C*UL)/U
      EH(NJ2-1)=E(NJ2)+(1.-P(NJ2))*EH(NJ2)
      XV(NJ2-1)=EH(NJ2-1)-(VJ2+1.)*C
  103 WRITE(3,4) J2,P(NJ2),E(NJ2),EH(NJ2-1),XV(NJ2-1)
      WRITE(3,5)
      WRITE(3,6) EH(1)
      WRITE(3,7)E(N)
      WRITE(3,2)
C     CALCULATION OF EXPECTED ITEM COST FOR THE NEW DISTRIBUTION USING
C     THE INITIAL PRICE BREAKS
      DO 104 J2=1,N2
      NJ2=N-J2
      IF(X(NJ2)-UL)32,32,33
   32 P(NJ2)=0.0
      E(NJ2)=0.0
      GO TO 34
   33 P(NJ2)=(X(NJ2)-UL)/U
      VJ2=J2
      E(NJ2)=((X(NJ2)*X(NJ2))/2.0+VJ2*C*X(NJ2)-(UL*UL)/2.0-VJ2*C*UL)/U
   34 EH(NJ2-1)=E(NJ2)+(1.-P(NJ2))*EH(NJ2)
  104 CONTINUE
      WRITE(3,8) EH(1)
   31 CONTINUE
      CALL EXIT
      END
```

Error in the Estimated Mean Cost - Computer Program (Uniform Distribution)

Appendix 4-A

```
C     SENSITIVITY ANALYSIS  1 -VARIATIONS IN PRICE BREAKS OF UNIF. DIST.
      DIMENSION E(90),EH(90),X(90),P(90),XV(90),PERC(51)
   21 FORMAT(I2)
    1 FORMAT(14F5.0)
   24 FORMAT (6F10.0,F5.0)
   25 FORMAT(1H1,36HINVENTORY MODEL USING BAYES CRITERIA//)
   26 FORMAT(1H ,8HLOTS(XS),16X,4HCOST,8X,6HPERIOD,10X,12HDAILY CHARGE/)
   27 FORMAT(1H ,F10.4,4X,E14.8,4X,F10.4,4X,I4,4X,F10.5////)
    2 FORMAT (1H ,4X,2HJ2,12X,1HP,18X,1HE,17X,2HEH,18X,1HX//)
    3 FORMAT(1H ,5X,1H0,18X,E14.8, 5X,E14.8, 5X,E14.8/)
    4 FORMAT(1H ,3X,I3,3X,F10.7, 5X,E14.8, 5X,E14.8, 5X,E14.8/)
    5 FORMAT(1H ,30HDISREGARD THE LAST PRICE BREAK//)
    6 FORMAT(' EXPECTED UNIT COST WITH MODIFIED PRICE BREAKS',3X,E14.8/)
    7 FORMAT(' EXPECTED UNIT COST W/O PRICE BREAKS',3X,E14.8///)
   11 FORMAT(' INV SYS W PRICE BREAKS=OPT BREAKS +',F5.2,'*STD DEV'//)
C     CALCULATION OF PRICE BREAKS AND EXPECTED ITEM COST
      READ(2,21)NP
C     NP=NUMBER OF POINTS DESIRED
      READ(2,1) (PERC(I),I=1,NP)
   23 READ(2,24)U,UL,UP,DN,D,T,CI
      WRITE(3,25)
      XS=((DN*CI*(UL+UP)/2.+DN*T)/(2.*D))**(0.5)
      AIP=365./XS
      IPER=AIP
      C=(CI*(UL+UP)/2.0+T)/365.0
      Y=DN*(UL+UP)/2.0+ D*XS+ (DN*(UL+UP)*CI)/(4.0*XS)+ (DN*T)/(2.0*XS)
      WRITE(3,26)
      WRITE(3,27)XS,Y,AIP,IPER,C
      I=1
      WRITE(3,11)PERC(I)
      WRITE(3,2)
      N=IPER+1
      E(N)=(UL+UP)/2.0
      AVG=E(N)
      EH(N-1)=E(N)
      X(N-1)=EH(N-1)-1.*C
```

```
      WRITE(3,3) E(N),EH(N-1),X(N-1)
      N2=N-2
      DO 102 J2=1,N2
      NJ2=N-J2
      P(NJ2)=(X(NJ2)-UL)/U
      VJ2=J2
      E(NJ2)=((X(NJ2)*X(NJ2))/2.0+VJ2*C*X(NJ2)-(UL*UL)/2.0-VJ2*C*UL)/U
      EH(NJ2-1)=E(NJ2)+(1.-P(NJ2))*EH(NJ2)
      X(NJ2-1)=EH(NJ2-1)-(VJ2+1.0)*C
  102 WRITE(3,4) J2,P(NJ2),E(NJ2),EH(NJ2-1),X(NJ2-1)
      WRITE(3,5)
      WRITE(3,6)EH(1)
      WRITE(3,7)AVG
C     CALCULATION OF VARIANCE AND STANDARD DEVIATION
      VAR=(U*U)/12.
      STDEV=VAR**(0.5)
      DO 31  I=2,NP
      WRITE(3,11) PERC(I)
      LM=N-1
C     CALCULATION OF DIFFERENT PRICE BREAKS
      DO 32 J=1,LM
   32 XV(J)=X(J)+PERC(I)*STDEV
C     CALCULATION OF EXPECTED ITEM COSTS USING NON-OPTIMUM PRICE BREAKS
      DO 103 J2=1,N2
      NJ2=N-J2
      IF(XV(NJ2)-UL)33,33,34
   33 P(NJ2)=0.0
      E(NJ2)=0.0
      GO TO 35
   34 P(NJ2)=(XV(NJ2)-UL)/U
      VJ2=J2
      E(NJ2)=((XV(NJ2)**2.)/2. +VJ2*C*XV(NJ2)-(UL*UL)/2.-VJ2*C*UL)/U
   35 EH(NJ2-1)=E(NJ2)+(1.-P(NJ2))*EH(NJ2)
  103 CONTINUE
      WRITE(3,6) EH(1)
      WRITE(3,7)AVG
```

```
31 CONTINUE
   CALL EXIT
   END
```

Error in the Price Breaks Used - Computer Program (Uniform Distribution)

Appendix 4-B

APPENDIX 5

Deviation from Optimality - Error in the Cost Probability Function Assumed

A - Computer Program - Calculation of Price Breaks and Expected Cost Assuming a Normal Distribution

B - Computer Program - Calculation of Expected Cost Assuming a Normal Distribution and Using the Price Breaks from a Uniform Distribution

```
C     PRICE BREAKS FOR OPTIMUM PURCHASING POLICIES IN AN INVENTORY SYS.(NORMAL)
      DIMENSION E(90),EH(90),X(90),P(90),PN(300),HN(300)
  211 FORMAT(F4.4)
   24 FORMAT(5F10.0,F5.0)
   25 FORMAT(1H1,36HINVENTORY MODEL USING BAYES CRITERIA///)
   26 FORMAT(1H ,8HLOTS(XS),16X,4HCOST,8X,6HPERIOD,10X,12HDAILY CHARGE/)
   27 FORMAT(1H ,F10.4,4X,E14.8,4X,F10.4,4X,I4,4X,F10.5///)
    2 FORMAT (1H ,4X,2HJ2,12X,1HP,18X,1HE,17X,2HEH,18X,1HX//)
    3 FORMAT(1H ,5X,1HO,18X,E14.8, 5X,E14.8, 5X,E14.8/)
    4 FORMAT(1H ,3X,I3,3X,F10.7, 5X,E14.8, 5X,E14.8, 5X,E14.8/)
    5 FORMAT(1H ,30HDISREGARD THE LAST PRICE BREAK//)
    6 FORMAT(1H ,37HEXPECTED TOTAL COST WITH PRICE BREAKS,3X,E14.8)
   11 FORMAT(1H ,32HPRICE BREAKS AND EXPECTED VALUES///)
   99 FORMAT(' ')
   15 FORMAT(34HEXPECTED PRICE FOR PURCHASE PERIOD,E14.8,2X,I5,2X,F10.3)
   12 FORMAT(10HDAYS LEFT=,I3,7X,E14.8)
C     READ NORMAL PROBABILITIES
      DO 210 IN=1,300
  210 READ(2,211) PN(IN)
C     READ NORMAL ORDINATES
      DO 212 JN=1,300
  212 READ(2,211) HN(JN)
C     READ COST DISTRIBUTION AND INVENTORY PARAMETERS
      READ(2,24)UM,USD,DN,D,T,CI
      WRITE(3,25)
C     CALCULATE STANDARD INVENTORY FIGURES
      XS=((DN*CI*UM+DN*T)/(2.*D))**(0.5)
      AIP=365./XS
C     IPER=NUMBER OF PERIODS IN THE CYCLE
      IPER=AIP
      C=(CI*UM+T)/365.0
      Y=DN*UM+D*XS+(DN*UM*CI)/(2.0*XS)+(DN*T)/(2.0*XS)
      WRITE(3,26)
      WRITE(3,27)XS,Y,AIP,IPER,C
   50 WRITE(3,11)
      WRITE(3,2)
```

```
      N=IPER+1
C     START APPLICATION OF THE PRICE BREAK MODEL
      E(N)=UM
      EH(N-1)=E(N)
      X(N-1)=EH(N-1)-1.*C
      WRITE(3,3) E(N),EH(N-1),X(N-1)
      N2=N-2
      DO 102 J2=1,N2
      NJ2=N-J2
      XNOR=-(X(NJ2)-UM)/USD
      XNOR1=XNOR*100.
      IXNOR=XNOR1
      XNOR2=IXNOR
      RES=XNOR1-XNOR2
      IF(RES-.5) 201,202,202
  202 IXNOR=IXNOR+1
  201 CONTINUE
      IF(IXNOR) 203,203,204
  203 P(NJ2)=.5
      VJ2=J2
      E(NJ2)=UM*P(NJ2)-USD*.3989+VJ2*C*P(NJ2)
      GO TO 205
  204 P(NJ2)=PN(IXNOR)
      VJ2=J2
      E(NJ2)=UM*P(NJ2)-USD*HN(IXNOR)+VJ2*C*P(NJ2)
  205 CONTINUE
      EH(NJ2-1)=E(NJ2)+(1.-P(NJ2))*EH(NJ2)
      X(NJ2-1)=EH(NJ2-1)-(VJ2+1.0)*C
  102 WRITE(3,4) J2,P(NJ2),E(NJ2),EH(NJ2-1),X(NJ2-1)
      WRITE(3,5)
C     EXVA=EXPECTED UNIT COST USING THE MODEL
      EXVA=EH(1)
      Y=DN*EXVA +D*XS +(DN*EXVA*CI)/(2.0*XS) +(DN*T)/(2.0*XS)
      WRITE(3,6) Y
      WRITE(2,99)
      WRITE(2,15) EH(1),IPER,AIP
```

```
      N1=N-1
      DO 14 I=1,N1
      L=N-I
   14 WRITE(2,12) I,X(L)
      CALL EXIT
      END
```

Computer Program - Calculation of Price Breaks and Expected Cost Assuming a Normal Distribution

Appendix 5-A

```
C     PRICE BREAKS FOR OPTIMUM PURCHASING POLICIES IN AN INVENTORY SYS.
C     NORMAL COST DISTRIBUTION WITH PRICE BREAKS CALCULATED ASSUMING
C     A UNIFORM DISTRIBUTION
      DIMENSION E(90),EH(90),X(90),P(90),PN(300),HN(300)
  211 FORMAT(F4.4)
   24 FORMAT(5F10.0,F5.0)
   25 FORMAT(1H1,36HINVENTORY MODEL USING BAYES CRITERIA///)
   26 FORMAT(1H ,8HLOTS(XS),16X,4HCOST,8X,6HPERIOD,10X,12HDAILY CHARGE/)
   27 FORMAT(1H ,F10.4,4X,E14.8,4X,F10.4,4X,I4,4X,F10.5///)
    2 FORMAT (1H ,4X,2HJ2,12X,1HP,18X,1HE,17X,2HEH,18X,1HX//)
    3 FORMAT(1H ,5X,1H0,18X,E14.8, 5X,E14.8, 5X,E14.8/)
    4 FORMAT(1H ,3X,I3,3X,F10.7, 5X,E14.8, 5X,E14.8, 5X,E14.8/)
    5 FORMAT(1H ,30HDISREGARD THE LAST PRICE BREAK//)
    6 FORMAT(1H ,37HEXPECTED TOTAL COST WITH PRICE BREAKS,3X,E14.8)
   11 FORMAT(1H ,32HPRICE BREAKS AND EXPECTED VALUES///)
   99 FORMAT(' ')
   15 FORMAT(34HEXPECTED PRICE FOR PURCHASE PERIOD,E14.8,2X,I5,2X,F10.3)
   12 FORMAT(10HDAYS LEFT=,I3,7X,E14.8)
  220 FORMAT(34X,E14.8,2X,I5,2X,F10.3)
  222 FORMAT(20X,E14.8)
C     READ NORMAL PROBABILITIES
      DO 210 IN=1,300
  210 READ(2,211) PN(IN)
C     READ NORMAL ORDINATES
      DO 212 JN=1,300
  212 READ(2,211) HN(JN)
C     READ MODEL RESULTS ASSUMINGA UNIFORM DISTRIBUTION OF COSTS
      READ(2,220) EXVAU,IPERU,AIPU
      NU=IPERU-1
C     READ PRICE BREAKS CALCULATED ASSUMING A UNIFORM COST DISTRIBUTION
      DO 221 I=1,NU
  221 READ(2,222) X(I)
C     READ COST DISTRIBUTION AND INVENTORY PARAMETERS
      READ(2,24)UM,USD,DN,D,T,CI
      WRITE(3,25)
C     CALCULATE STANDARD INVENTORY FIGURES
```

```
      XS=((DN*CI*UM+DN*T)/(2.*D))**(0.5)
      AIP=365./XS
      IPER=AIP
      C=(CI*UM+T)/365.0
      Y=DN*UM+D*XS+(DN*UM*CI)/(2.0*XS)+(DN*T)/(2.0*XS)
      WRITE(3,26)
      WRITE(3,27)XS,Y,AIP,IPER,C
   50 WRITE(3,11)
      WRITE(3,2)
      N=IPER+1
C     APPLICATION OF PRICE BREAK MODEL WHEN THE COST DISTRIBUTION IS
C     NORMAL AND THE PRICE BREAKS WERE CALCULATED ASSUMING A UNIFORM
C     DISTRIBUTION
      E(N)=UM
      EH(N-1)=E(N)
      WRITE(3,3) E(N),EH(N-1),X(N-1)
      N2=N-2
      DO 102 J2=1,N2
      NJ2=N-J2
      XNOR=-(X(NJ2)-UM)/USD
      XNOR1=XNOR*100.
      IXNOR=XNOR1
      XNOR2=IXNOR
      RES=XNOR1-XNOR2
      IF(RES-.5) 201,202,202
  202 IXNOR=IXNOR+1
  201 CONTINUE
      IF(IXNOR) 203,203,204
  203 P(NJ2)=.5
      VJ2=J2
      E(NJ2)=UM*P(NJ2)-USD*.3989+VJ2*C*P(NJ2)
      GO TO 205
  204 P(NJ2)=PN(IXNOR)
      VJ2=J2
      E(NJ2)=UM*P(NJ2)-USD*HN(IXNOR)+VJ2*C*P(NJ2)
  205 CONTINUE
```

```
      EH(NJ2-1)=E(NJ2)+(1.-P(NJ2))*EH(NJ2)
  102 WRITE(3,4) J2,P(NJ2),E(NJ2),EH(NJ2-1),X(NJ2-1)
      WRITE(3,5)
C     EXVA=EXPECTED UNIT COST USING THE MODEL WITH THE WRONG
C     DISTRIBUTION ASSUMED
      EXVA=EH(1)
      Y=DN*EXVA +D*XS +(DN*EXVA*CI)/(2.0*XS) +(DN*T)/(2.0*XS)
      WRITE(3,6) Y
      WRITE(2,99)
      WRITE(2,15) EH(1),IPER,AIP
      N1=N-1
      DO 14 I=1,N1
      L=N-I
   14 WRITE(2,12) I,X(L)
      CALL EXIT
      END
```


Computer Program - Calculation of Expected Cost Assuming a Normal Distribution and Using the Price Breaks from a Uniform Distribution

Appendix 5-B

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Educational Background

--Instituto de Matanzas (High School)	Graduated 1956
--Georgia Institute of Technology Bachelor of Science in Electrical Engineering (with Highest Honors)	Graduated 1960
Master of Science in Electrical Engineering	Attended 1963-1965
--Lehigh University Candidate for Master of Science Degree, Industrial Engineering	1967

Honors

--Phi Eta Sigma
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--Tau Beta Pi
--Phi Kappa Phi
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--Selected by Eta Kappa Nu as the outstanding student in the Electrical Engineering department of Georgia Institute of Technology in 1960.
--Tau Beta Pi cup to the outstanding engineering senior in Georgia Institute of Technology in 1960.

Professional Experience

Cuban Electric Power Company
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Engineering Associate - Distribution and Transmission
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Professional Experience (Cont.)

Industrial Ministry
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Project Engineer - Process improvements, methods, special studies.
June 1960 - July 1962

Western Electric Co., Inc.
Atlanta, Georgia
Equipment Engineer
November 1962-July 1965

Western Electric Co., Inc.
Princeton, New Jersey
Research Engineer, Lehigh Master's Program
July 1965 -

Professional Organizations

Institute of Electrical and Electronics Engineers
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Engineer-in-Training - State of Georgia